Thermal properties of interacting Bose fields and imaginary-time stochastic differential equations

L. I. PLIMAK[1], M. FLEISCHHAUER[2] and D. F. WALLS[1]

[1] *Department of Physics, University of Auckland*
Private Bag 92019, Auckland, New Zealand
[2] *Sektion Physik, Ludwig-Maximilians Universität München*
D-80333 München, Germany

Abstract. – Matsubara Green's functions for interacting bosons are expressed as classical statistical averages corresponding to a linear imaginary-time stochastic differential equation. This makes direct numerical simulations applicable to the study of equilibrium quantum properties of bosons in the non-perturbative regime. To verify our results we discuss an oscillator with quartic anharmonicity as a prototype model for an interacting Bose gas. An analytic expression for the characteristic function in a thermal state is derived and a Higgs-type phase transition discussed, which occurs when the oscillator frequency becomes negative.

Introduction. – A commonly used method to study thermal properties of interacting many-body systems is the Matsubara Green's function technique [1-3]. In this paper, we present a method of evaluating Matsubara Green's functions for bosons, based on constructively characterising Feynman paths as solutions to an Ito stochastic differential equation (SDE). (For stochastic calculus and SDEs see [4]; a brief discussion is given below.) Numerically simulating this SDE then allows one to calculate physical quantities using their expressions as path integrals [5] (this constitutes a constructive rather than a Monte Carlo method [6]). Since path integrals correspond to an in principle *exact summation* of the Matsubara diagram series as a whole (as opposed to approximate *partial summations* underlying conventional Green's function approaches [2,3]), the proposed technique is applicable beyond the perturbation region and is only limited by numerical errors. It could be advantageous, *e.g.*, for analysing the behaviour of an interacting Bose gas near the critical temperature of condensation [7].

We develop a version of the Matsubara technique where a diagram series is averaged over the thermal P-distribution for free bosons [8]. For the grand-canonical ensemble this distribution is positive and hence can be interpreted as a probability density. Then we see that summing the series is equivalent to solving a certain SDE. For bosons with quartic interaction, this SDE is a linear imaginary-time Schrödinger equation with multiplicative noise. As a result, we express

normally ordered thermal averages of the bosonic field as classical statistical averages. We verify this result, including the choice of stochastic calculus (Ito), by considering a prototype model of interacting Bose fields —a quantized oscillator with quartic anharmonicity. A simple analytic expression for the characteristic function is derived and a Higgs-type phase transition discussed, which occurs when the frequency of the oscillator becomes negative.

General theory. – Consider a system of interacting Bose fields in an external potential $V = V(\vec{r})$, described by the Schrödinger field operator $\hat{\psi} = \hat{\psi}(\vec{r})$, with the Hamiltonian $H = H_0 + H_{\text{int}}$, where

$$H_0 = \int d^3\vec{r}\,\hat{\psi}^\dagger \left(-\frac{1}{2m}\nabla^2 + V \right) \hat{\psi}, \qquad H_{\text{int}} = H_{\text{int}}(\hat{\psi}^\dagger, \hat{\psi}) = \frac{\kappa}{2} \int d^3\vec{r}\,\hat{\psi}^{\dagger 2}\hat{\psi}^2, \qquad (1)$$

$\kappa = 4\pi a/m$ and a is the s-wave scattering length. We use units where $\hbar = 1$.

The aim of the paper is to calculate thermal averages of normally ordered products of $\hat{\psi}$ and $\hat{\psi}^\dagger$ over the grand-canonical density matrix, $\rho = e^{(\Omega - H + \mu N)\beta}$. Here $\beta = 1/k_{\text{B}}T$, T is the temperature, μ is the chemical potential, and $Z = \exp[-\Omega\beta]$ is the partition function. To this end, we consider the normally ordered characteristic functional

$$\chi(\xi, \xi^*) = \left\langle \exp\left[\xi\hat{\psi}^\dagger \right] \exp\left[-\xi^*\hat{\psi} \right] \right\rangle, \qquad (2)$$

where $\xi\hat{\psi}^\dagger \equiv \int d^3\vec{r}\,\xi(\vec{r})\hat{\psi}^\dagger(\vec{r})$ (say) and $\xi = \xi(\vec{r})$ is an arbitrary complex function.

To relate averages over ρ to those over the density matrix for the free system, $\rho_0 = e^{(\Omega_0 - H_0 + \mu N)\beta}$, we introduce the Matsubara interaction picture

$$\hat{\psi}_T(\vec{r}, \tau) = e^{(H_0 - \mu N)\tau}\hat{\psi}e^{-(H_0 - \mu N)\tau}, \qquad (3)$$

$$\hat{\bar{\psi}}_T(\vec{r}, \tau) = e^{(H_0 - \mu N)\tau}\hat{\psi}^\dagger e^{-(H_0 - \mu N)\tau} = \hat{\psi}_T^\dagger(\vec{r}, -\tau), \qquad (4)$$

and the thermal "scattering matrix"

$$\mathcal{S} = T_\tau\exp\left[-\mathcal{H}(\hat{\bar{\psi}}_T, \hat{\psi}_T) \right], \qquad (5)$$

where $\mathcal{H}(\hat{\bar{\psi}}_T, \hat{\psi}_T) = \int_{-\beta}^0 d\tau H_{\text{int}}(\hat{\bar{\psi}}_T, \hat{\psi}_T)$, and T_τ is the "time"-ordering operator with respect to τ. (Note that we consider Matsubara evolution from $\tau = -\beta$ to $\tau = 0$, not from 0 to β as usual: $\tau = 0$ is the latest, not the earliest, "time"; some of our relations therefore differ slightly from those in the literature.) Using $\rho = \mathcal{S}\rho_0/\langle\mathcal{S}\rangle_0$, where $\langle\ldots\rangle_0$ denotes the "free" averaging over ρ_0, we find with $\tilde{\chi}(\xi, \xi^*) \equiv \chi(\xi, \xi^*)\langle\mathcal{S}\rangle_0$

$$\tilde{\chi}(\xi, \xi^*) = \left\langle T_\tau\exp\left[-\mathcal{H}(\hat{\bar{\psi}}_T, \hat{\psi}_T) + \xi\hat{\bar{\psi}}_T(+0) - \xi^*\hat{\psi}_T(-0) \right] \right\rangle_0. \qquad (6)$$

Matsubara diagram series [1-3] are obtained by expanding the exponent in (6) in a power series, changing the order of the summation and averaging, and then factorising the many-operator averages into products of free thermal Green's functions [3,9]. Physical quantities are then obtained isolating certain "leading" classes of diagrams which may be summed yielding Dyson equations. This inevitably involves approximations which fail if the interaction is not small or in the vicinity of a phase transition. In order to have an approach which is in principle exact, we here proceed in a different way. We first apply Wick's theorem proper so as to bring the operator expression in (6) to normal order, and then employ the relation

$$\left\langle : F(\hat{\psi}^\dagger, \hat{\psi}) : \right\rangle_0 = \int D^2\psi_0\, P(\psi_0(\vec{r}))\langle\psi_0|: F(\hat{\psi}^\dagger, \hat{\psi}) :|\psi_0\rangle \equiv \overline{F(\psi_0^*, \psi_0)}. \qquad (7)$$

Here, F is an arbitrary functional of $\hat{\psi}(\vec{r})$ and $\hat{\psi}^\dagger(\vec{r})$, $\int D^2\psi_0$ is a functional integration and $P(\psi_0)$ is the Glauber P-distribution for the non-interacting bosons in a thermal state, which is positive [8]. Wick's theorem can be given a compact form using functional derivatives [10,11]

$$T_\tau F(\hat{\tilde{\psi}}_T, \hat{\psi}_T) = \; : e^\Delta F(\tilde{\phi}, \phi)\big|_{\phi \to \hat{\psi}_T; \tilde{\phi} \to \hat{\tilde{\psi}}_T} \; : \; , \tag{8}$$

where

$$\Delta = \iint_{-\beta}^0 d\tau_1 d\tau_2 \iint d^3\vec{r}_1 d^3\vec{r}_2 \, D_T(\vec{r}_1, \vec{r}_2; \tau_1 - \tau_2) \frac{\delta^2}{\delta\phi(\vec{r}_1, \tau_1)\delta\tilde{\phi}(\vec{r}_2, \tau_2)}, \tag{9}$$

$\phi(\vec{r}, \tau)$ and $\tilde{\phi}(\vec{r}, \tau)$ are arbitrary independent c-number functions, and

$$D_T(\vec{r}_1, \vec{r}_2; \tau_1 - \tau_2) = \left\langle 0 \left| T_\tau \hat{\psi}_T(\vec{r}_1, \tau_1)\hat{\tilde{\psi}}_T(\vec{r}_2, \tau_2) \right| 0 \right\rangle = \theta(\tau_1 - \tau_2) \left[\hat{\psi}_T(\vec{r}_1, \tau_1), \hat{\tilde{\psi}}_T(\vec{r}_2, \tau_2) \right] \tag{10}$$

is the Matsubara contraction. Note that eq. (8) does not refer to any state vector and that D_T in (10) is a vacuum and not a thermal average: as opposed to the usual Matsubara diagram series the thermal properties of the non-interacting system are contained in the initial distribution $P(\psi_0)$ and not in the propagator. It is straightforward to prove that at the same time D_T is the retarded Green's function of the imaginary-time free Schrödinger equation:

$$\left(\frac{\partial}{\partial\tau_1} - \frac{1}{2m}\nabla_1^2 + V(\vec{r}_1) - \mu \right) D_T(\vec{r}_1, \vec{r}_2; \tau_1 - \tau_2) = \delta(\tau_1 - \tau_2)\delta^3(\vec{r}_1 - \vec{r}_2). \tag{11}$$

Applying (8) to (6), and using (7), we find

$$\tilde{\chi}(\xi, \xi^*) = \overline{\Phi}, \quad \Phi = e^\Delta \exp\left[-\mathcal{H}(\tilde{\phi}, \phi) + \xi\tilde{\phi}(0) - \xi^*\phi(0)\right]\bigg|_{\phi \to \psi_0; \tilde{\phi} \to \tilde{\psi}_0} . \tag{12}$$

Here $\psi_0 = \psi_0(\vec{r}, \tau)$ and $\tilde{\psi}_0 = \tilde{\psi}_0(\vec{r}, \tau) = \psi_0^*(\vec{r}, -\tau)$ are the amplitudes of the non-interacting fields in the coherent state $|\psi_0\rangle$: $\hat{\psi}_T(\vec{r}, \tau)|\psi_0\rangle = \psi_0(\vec{r}, \tau)|\psi_0\rangle$ and $\langle\psi_0|\hat{\tilde{\psi}}_T(\vec{r}, \tau) = \langle\psi_0|\tilde{\psi}_0(\vec{r}, \tau)$. For $\tau = 0$ the Matsubara interaction picture coincides with the Schrödinger picture, hence $\psi_0(\vec{r}, 0) = \psi_0(\vec{r})$ and $\tilde{\psi}_0(\vec{r}, 0) = \psi_0^*(\vec{r})$.

Our goal is now to express (12) as a classical statistical average. Since $P(\psi_0) \geq 0$, this interpretation applies to the averaging over ψ_0 (denoted by the upper bar in (12)). In the quantity Φ one can replace $\xi\tilde{\phi}(0)$ by $\xi\psi_0^*$, since $\tau = 0$ is the largest "time" and the exponential operator does not act on $\tilde{\phi}(0)$. Following the Stratonovich-Hubbard transformation used in path-integral approaches [12], we introduce a Gaussian stochastic variable $\eta(\vec{r}, \tau)$, such that $\overline{\eta(\vec{r}, \tau)\eta(\vec{r}', \tau')} = \kappa\,\delta^3(\vec{r} - \vec{r}')\delta(\tau - \tau')$, and write

$$\exp\left[-\mathcal{H}(\tilde{\phi}, \phi)\right] = \overline{\exp\left[i\tilde{\phi}\eta\phi\right]}, \tag{13}$$

where $\tilde{\phi}\eta\phi = \int_{-\beta}^0 d\tau \int d^3\vec{r} \; \tilde{\phi}(\vec{r}, \tau)\,\eta(\vec{r}, \tau)\,\phi(\vec{r}, \tau)$. We then show that

$$e^\Delta \exp\left[i\tilde{\phi}\eta\phi - \xi^*\phi(0)\right]\bigg|_{\phi \to \psi_0; \tilde{\phi} \to \tilde{\psi}_0} = \exp\left[i\tilde{\psi}_0\eta\psi - \xi^*\psi(0)\right], \tag{14}$$

where $\psi = \psi(\vec{r}, \tau)$ is a solution of the Ito equation

$$\left(\frac{\partial}{\partial\tau} - \frac{1}{2m}\nabla^2 + V - \mu \right) \psi(\vec{r}, \tau) = i\eta(\vec{r}, \tau)\psi(\vec{r}, \tau), \tag{15}$$

with the initial condition given by

$$\psi(\vec{r}, -\beta) = \psi_0(\vec{r}, -\beta). \tag{16}$$

Indeed, expand all exponents on the LHS of (14) in power series and perform the functional differentiations. The emerging series is similar to a diagram series for a problem with a linear perturbation. In particular, all connected diagrams are linear chains, of the structure either $\tilde{\psi}_0 D_T i\eta D_T \cdots i\eta\psi_0$, or $-\xi^* D_T i\eta D_T \cdots i\eta\psi_0$. By virtue of Meyer's first theorem [12] this immediately results in the RHS of (14), with ψ obeying the Dyson equation

$$\psi(\vec{r},\tau) = \psi_0(\vec{r},\tau) + \int\mathrm{d}^3\vec{r}' \int_{-\beta}^0 \mathrm{d}\tau'\, D_T(\vec{r},\vec{r}',\tau-\tau')i\eta(\vec{r}',\tau')\psi(\vec{r}',\tau'), \qquad (17)$$

which in turn is equivalent to (15).

For $\eta = 0$, eq. (15) describes the evolution of ψ_0, while changing $\tau \to -\tau$ yields the equation for $\tilde{\psi}_0$. Hence $\frac{\partial}{\partial\tau}\int\mathrm{d}^3\vec{r}\,\tilde{\psi}_0(\vec{r},\tau)\psi(\vec{r},\tau) = i\int\mathrm{d}^3\vec{r}\,\tilde{\psi}_0(\vec{r},\tau)\eta(\vec{r},\tau)\psi(\vec{r},\tau)$ and we find that

$$i\tilde{\psi}_0\eta\psi \equiv i\int_{-\beta}^0\mathrm{d}\tau\int\mathrm{d}^3\vec{r}\,\tilde{\psi}_0(\vec{r},\tau)\eta(\vec{r},\tau)\psi(\vec{r},\tau) = \int\mathrm{d}^3\vec{r}\,\psi_0^*(\vec{r})\left[\psi(\vec{r},0) - \psi_0(\vec{r})\right].$$

Finally,

$$\chi(\xi,\xi^*) = \tilde{\chi}(\xi,\xi^*)/\tilde{\chi}(0,0), \quad \tilde{\chi}(\xi,\xi^*) = \overline{\overline{\mathrm{e}^\Lambda}}, \qquad (18)$$

$$\Lambda = \int\mathrm{d}^3\vec{r}\left[\xi(\vec{r})\psi_0^*(\vec{r}) - \xi^*(\vec{r})\psi(\vec{r},0) + \psi_0^*(\vec{r})\psi(\vec{r},0) - |\psi_0(\vec{r})|^2\right]. \qquad (19)$$

The double upper bar in (18) denotes an averaging over both $\psi_0(\vec{r})$ and $\eta(\vec{r},\tau)$; *i.e.* the statistics here is that of the random trajectories ($\equiv$ Feynman paths) $\psi(\vec{r},\tau)$, which are solutions to the SDE (15), with the random initial condition distributed with probability $P(\psi_0)$. The interaction between the bosons is contained in the noise η, while the thermal properties of the system in $P(\psi_0)$. Relation (18) is the main result of this paper.

It is important that its derivation implies a *causal regularisation* of the function D_T. Namely, D_T (which is a generalised function) is replaced by a certain number of times continuously differentiable function, yet preserving the causality condition $D_T(\vec{r},\vec{r}',\tau) = 0, \tau < 0$. Consider, for example, relation (12). Expanding all exponents in (12) and performing the functional differentiations results in the quantity Φ being expressed as a diagram series. We see that the causal regularisation is exactly what is necessary to eliminate the "short-circuited" diagrams (containing $D_T(\vec{r},\vec{r},0)$ which is undefined), giving rise to infinities. These diagrams emerge when the differential operator Δ is applied to a single $\mathcal{H}$, whereas their absence is required by the normal form of the interaction (no contractions within the same interaction Hamiltonian). Note that this regularisation makes sense of the replacement $\xi\tilde{\phi}(0) \to \xi\psi_0^*$ we used above.

The causal regularisation of D_T is also what leads to the interpretation of (15) as an Ito SDE. Since the noise source $\eta(\vec{r},\tau)$ in (15) is singular (delta-correlated white noise), the SDE is, mathematically speaking, not defined. However, the corresponding integral equation can consistently be interpreted [4] with a proper definition of the stochastic increment, $\psi(\tau)\eta(\tau)\mathrm{d}\tau \equiv \psi(\tau)\,\mathrm{d}W$, where $W(\tau) = \int^\tau \eta(\tau')\mathrm{d}\tau'$. In the Ito stochastic calculus $\psi(\tau)\mathrm{d}W$ is understood as $\psi(\tau)[W(\tau+\mathrm{d}\tau) - W(\tau)]$, while in Stratonovich calculus it is $\frac{1}{2}[\psi(\tau) + \psi(\tau+\mathrm{d}\tau)][W(\tau+\mathrm{d}\tau) - W(\tau)]$. Alternatively [13], one may replace (15) by the corresponding integral equation, $\psi = \psi_0 + iD_T\eta$, which is defined given D_T is regularised. It is then easy to see that the causal regularisation of D_T results in $\psi(\tau)$ being uncorrelated with $\eta(\tau)$, which is the characteristic property of the Ito calculus. More detailed arguments in favour of this conjecture are given in [13].

It is worth noting that the diagram series for Φ is structurally identical to that for Bose-condensed systems [2,3], where the propagator is replaced by D_T and the condensate amplitude

by the coherent amplitude ψ_0. Yet, despite this formal similarity, these series are clearly distinct. Here the coherent amplitude ψ_0 applies to all bosonic states, not just the condensate. This makes it unnecessary to treat condensed and non-condensed fractions separately as in [2,3]. A novel feature is furthermore that our series as a whole and not the propagator is averaged over the free distribution $P(\psi_0)$. Consequently we find the propagator D_T equal to the vacuum average (10), whereas in [2,3] it is a thermal average.

Quantum oscillator. – To verify relation (18) including the said regularisation/calculus conjecture, consider an anharmonic oscillator described by annihilation and creation operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ with $H_0 = \omega_0 \hat{\psi}^\dagger \hat{\psi}$ and $H_{\text{int}} = \kappa/2 \, \hat{\psi}^{\dagger 2} \hat{\psi}^2$. The above results apply by simply dropping the spatial variable, $\psi(\vec{r}, \tau) \to \psi(\tau)$, etc. Then, $\psi_0(\tau) = \psi_0 e^{-\omega_0 \tau}$, $\tilde{\psi}_0(\tau) = \psi_0^* e^{\omega_0 \tau}$. The SDE, which now reads

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} - \omega_0 - i\eta(\tau) \right] \psi(\tau) = 0, \tag{20}$$

is readily solved [4], which yields $\psi(0) = z\psi_0$, where $z = se^{i\vartheta}$, $\vartheta = \int_{-\beta}^{0} \mathrm{d}\tau \, \eta(\tau)$ is the total random increment, $\overline{\vartheta^2} = \beta\kappa$, and $s = e^{\beta\kappa/2}$. This result corresponds to (20) regarded as an Ito equation. In Stratonovich calculus (say), one would find $s = 1$. Using the explicit expression for $P(\psi_0)$ [8], we have ($r = e^{\beta\omega_0}$)

$$\tilde{\chi}(\xi, \xi^*) = \int \frac{(r-1)e^{-(r-1)|\psi_0|^2} \mathrm{d}^2\psi_0}{\pi} \int \frac{e^{-\vartheta^2/2\beta\kappa} \mathrm{d}\vartheta}{\sqrt{2\pi\beta\kappa}} \exp\left[-|\psi_0|^2(1-z) + \xi\psi_0^* - \xi^* z\psi_0 \right]. \tag{21}$$

Changing the order of integrations in (21) and then taking the Gaussian integral over ψ_0 yields

$$\tilde{\chi}(\xi, \xi^*) = \overline{\frac{r-1}{r-z} \exp\left[-\frac{|\xi|^2 z}{r-z} \right]} = \frac{r-1}{r} \sum_{n=0}^{\infty} L_n(|\xi|^2) \left(\frac{s}{r} \right)^n e^{-n^2 \beta\kappa/2}, \tag{22}$$

where the upper bar denotes the integration (averaging) over ϑ and L_n are the Laguerre polynomials.

To verify the choice of the Ito calculus consider the partition function of the anharmonic oscillator. Noting that $\tilde{\chi}(0,0) = \langle S \rangle_0 = Z/Z_0$, $Z_0 = 1/(1 - e^{-\beta\omega_0}) = r/(r-1)$, and using (22)
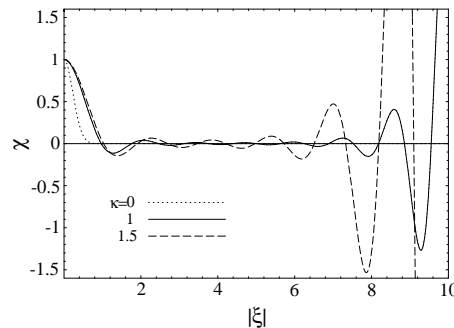

Fig. 1. – Characteristic function of the oscillator for $\omega_0 = 1$, $\beta = 0.1$ and different values of anharmonicity.
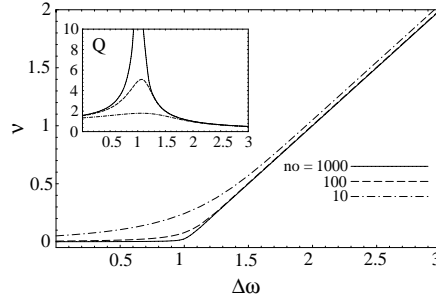
Fig. 2. – Higgs-type phase transition in the anharmonic oscillator. Mean number of quanta and fluctuations (insert) for $\omega_0 = 1$, $\beta = 1$ and different n_0.

we have

$$Z = \overline{\frac{1}{1 - z\mathrm{e}^{-\beta\omega_0}}} = \sum_{n=0}^{\infty} s^n \exp\left[-\beta\omega_0 n - \frac{\beta\kappa n^2}{2}\right]. \tag{23}$$

This should be compared to the Fock-space expansion, $Z = \sum_{n=0}^{\infty} \exp\left[-\beta\omega_0 n - \beta\kappa n(n-1)/2\right]$. We see that the "Ito-valued" $s = \mathrm{e}^{\beta\kappa/2}$ is exactly what is required to match (23) and this expression, while the "Stratonovich-valued" $s = 1$ results in a discrepancy.

In fig. 1, we plot the characteristic function for $\omega_0 = 1$, $\beta = 0.1$ and different values of the interaction constant. After an initial decay, χ exhibits exponentially growing oscillations. The P-function, which is the Fourier transform of χ, is hence not an ordinary function and the thermal state of the interacting system is *non-classical* as opposed to the non-interacting one.

The anharmonic oscillator shows an interesting behaviour if a negative linear part is added to the interaction, $H_{\mathrm{int}} = -\Delta\omega\,\hat{\psi}^\dagger\hat{\psi} + \kappa/2\,\hat{\psi}^{\dagger 2}\hat{\psi}^2$. The total linear part of the Hamiltonian, $(\omega_0 - \Delta\omega)\hat{n} = \omega\hat{n}$, may then be negative. If $\tilde{\psi}^2 = -\omega/\kappa > 0$, the ground state of the interacting system no longer coincides with that of the harmonic oscillator but with one of the Fock states. In the limit of infinite system size, $n_0 \equiv \omega_0/\kappa \to \infty$, the system undergoes a second-order Higgs-type phase transition, with $\tilde{\psi}$ being the order parameter. This is illustrated in fig. 2 which shows the normalized mean number of quanta $\nu = \langle\hat{n}\rangle/n_0$ and fluctuations
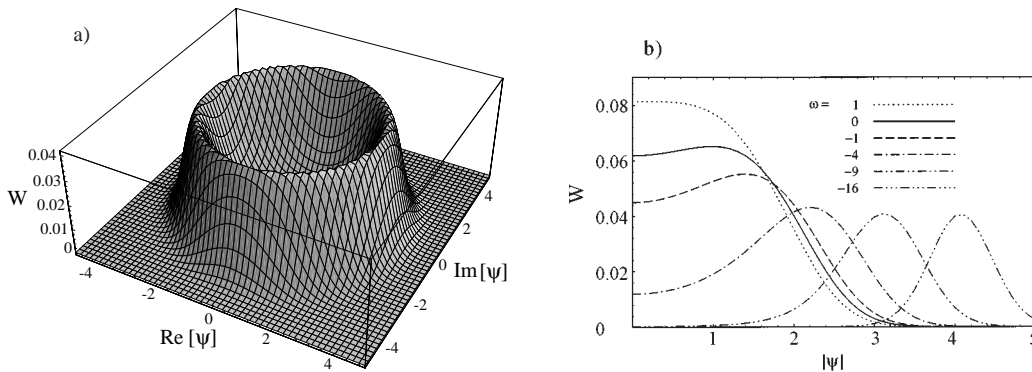

Fig. 3. – a) Typical shape of the Wigner function "above threshold" ($\beta = 0.1$, $\kappa = 1$, $\omega = -8$). b) The Wigner function for different values of ω.

$Q = \langle(\hat{n} - \langle\hat{n}\rangle)^2\rangle/\langle\hat{n}\rangle$ as a function of $\Delta\omega$ for different system-size parameters.

Above threshold, quantum distribution such as the Wigner function should become concentrated at $|\psi| \sim \tilde{\psi}$, rather than at $|\psi| \sim 0$. The above analysis is easily extended to include the additional linear interaction and all results apply with $s = \mathrm{e}^{\beta\kappa/2} \to s = \mathrm{e}^{\beta\Delta\omega+\beta\kappa/2}$. Thus

$$W(\psi, \psi^*) = \int \frac{\mathrm{d}^2\xi}{\pi^2}\, \mathrm{e}^{\xi\psi^*-\psi\xi^*-|\xi|^2/2}\chi(\xi, \xi^*) = \frac{2(r-1)}{\pi\tilde{\chi}(0,0)}\,\overline{\frac{1}{r+z}\exp\left[-2|\psi|^2\frac{r-z}{r+z}\right]}\ . \quad (24)$$

The averaging here is over ϑ. Rigorously the last equation in (24) holds only if $\Re(r-z) > 0$ and hence $\kappa < 2\omega$, but one can get rid of this condition by appropriately modifying the averaging (namely, moving it from the circle $|z| = s$ to $|z| = 1$ while preserving all $\overline{z^n}$, $n = 0, 1, \cdots$). The results of the corresponding numerical evaluation of the Wigner function shown in fig. 3 clearly indicate the phase transition when ω becomes negative.

In conclusion, we have shown that in the Feynman-path representation of the Matsubara Green's function of interacting bosons, the Feynman paths may be constructively characterised as solutions to an imaginary time Ito stochastic differential equation. This allows one to calculate normally ordered thermal averages of interacting nonrelativistic Bose fields beyond the level of perturbation. We have verified this result, including the fact that the equation we find is an Ito equation, for the simple prototype model of an anharmonic oscillator. The relation between stochastic calculus and regularisation as well as the application of the method to ultracold atomic quantum gases will be subject to further investigations.

$$***$$

REFERENCES

[1] MATSUBARA T., *Prog. Theor. Phys.*, **14** (1955) 351.
[2] ABRIKOSOV A. A., GORKOV L. P. and DZYALOSHINSKI I. E., *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall Inc., Englewood Cliffs, NJ) 1963; republished by Dover, New York, 1975.
[3] FETTER A. L. and WALECKA J. D., *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York) 1971.
[4] GARDINER C. W., *Handbook of Stochstic Methods* (Springer, Berlin) 1990.
[5] WIEGEL F. W., "Path Integral Methods in Statistical Mechanics", *Phys. Rep.*, **16** (1975) 57.
[6] CEPERLEY D. M., "Path integrals in the theory of condensed helium", *Rev. Mod. Phys.*, **67** (1995) 279.
[7] For a recent review on atomic Bose-condensates see: PARKINS A. S. and WALLS D. F., *The physics of trapped dilute-gas Bose-Einstein condensates*, to be published in *Phys. Rep.* (1998).
[8] GARDINER C. W., *Quantum Noise* (Springer, Berlin) 1991.
[9] DANIELEWICZ P., *Ann. Phys.*, **152** (1984) 239.
[10] AKHIEZER A. I. and BERESTETSKII V. B., *Elements of Quantum Electrodynamics* (Oldbourne Press, London) 1964.
[11] HORI T., *Prog. Theor. Phys.*, **7** (1952) 378.
[12] NEGERLE J. W. and ORLAND H., *Quantum Many-Particle Systems* (Addision Wesley, Reading, Mass.) 1978.
[13] PLIMAK L. I., FLEISCHHAUER M., COLLETT M. J. and WALLS D. F., preprint cond-mat/9712192.