

Comment on "Theory of growth of number entropy in disordered systems"

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We comment on the recent preprint by Ghosh and Žnidarič [arXiv:2112.12987] which studies particle fluctuations in the Heisenberg model with random magnetic fields after a quantum quench. The authors present arguments for an intermediate power-law growth in time of the particle fluctuations and a system-size independent saturation at long times, claiming consistency of their results with many-body localization (MBL) for strong disorder. Here we show that the unusual initial states considered in this work, which show only local initial dynamics, are not suitable to investigate MBL in small systems. Furthermore, we show that the power-law scaling for the number entropy is system-size dependent and does not hold for typical initial states with extensive energy fluctuations. We also argue that for the cases where the authors find that an effective resonance model works, the dynamics is of single particle type and unrelated to MBL.

I. INTRODUCTION

In a recent preprint [1], Ghosh and Žnidarič investigate quench dynamics in the Heisenberg model with random magnetic fields. They argue that for strong disorder and in the thermodynamic limit, the number entropy only grows as $S_N = \text{const} - B/t$ as a function of time t after the quench. This would be consistent with the existence of a many-body localized (MBL) phase. On the other hand, we have argued in a number of recent papers [2–6] that the growth of the von-Neumann entropy S and the number entropy S_N in this model are connected, $S_N \sim \ln S$, as in the non-interacting case albeit with renormalized coefficients. In particular, we have found that at strong disorder the von-Neumann entropy grows as $S \sim \ln t$ and the number entropy as $S_N \sim \ln \ln t$.

The authors of Ref. [1] reach their conclusions based on (a) exact diagonalization data for initial states which have so far not been considered to explore the possible existence of an MBL phase, and (b) a short-time theory based on single-particle resonances.

In this comment we will point out that: (i) Many of the initial states chosen in [1] are not suitable to investigate possible MBL physics in finite Heisenberg chains. (ii) The power law $S_N = \text{const} - B/t^\alpha$ with $\alpha = 1$ does not hold. Instead, the exponent α in such phenomenological fits strongly depends on system size and appears to go to zero in the thermodynamic limit. Also, the saturation value is not constant but rather system-size dependent as well. On the other hand, a $S_N = a \ln \ln t$ scaling does hold for different system sizes with the *same* prefactor a . (iii) The theory based on resonances is equally applicable to the von-Neumann entropy S . Indeed, for the initial states and large disorder strengths investigated in [1], the dynamics for the studied system sizes stays entirely local and $S \approx S_N$. Such behavior is not indicative of MBL physics but is rather fully explained by single particle physics.

II. CHOICE OF INITIAL STATE

It is important to stress that Anderson localization is not about the localization of a particle whose motion is already restricted to a finite region of space. Rather, the particle is in principle able to move infinitely far away from its initial position; what is stopping it from doing so for large potential disorder is that the effective coupling to another site falls off exponentially with distance while the energy mismatch only falls off like one over distance in one dimension [7]. Similarly, to study the possible existence of MBL one has to understand the occurrence of many-body resonances and avalanche instabilities which are global, not local, properties of the system [8].

To investigate the question whether or not an MBL phase in the Heisenberg model—which is equivalent to the spinless t-V fermion model by Jordan-Wigner transform—does exist, it is thus important to start from an initial state in which a macroscopic number of particles are able to move. I.e., initial states which do have *extensive* energy fluctuations ΔE^2 . This is the case, for example, for the Néel state and also for the vast majority of other product states. For this reason, almost all numerical studies of the MBL problem so far use the Néel state and/or randomly drawn product states as initial states. The authors of Ref. [1], on the other hand, choose very special product states as, for example, the domain wall state which has $O(1)$ energy fluctuations: only the particle at the edge of the domain wall is initially able to move. In particular, for all product states which are eigenstates of the total particle number operator \hat{N} the energy fluctuations can be simply quantified by the number of kinks in the state: $\Delta E^2 \sim N_{\text{kinks}}$. The Neel state has $N_{\text{kinks}} = L - 1$ and thus has extensive energy fluctuations while the domain wall state has $N_{\text{kinks}} = 1$ and therefore intensive energy fluctuations.

Another way to look at this issue is in terms of the difference between a global and a local quench: the domain wall state is an eigenstate of the Heisenberg model

with the bond connecting the two domains removed. So the quench is effectively a local one. Typical initial product states, on the other hand, are not eigenstates of the Hamiltonian with a finite number of local modifications. It should be noted, furthermore, that the domain wall state remains partially frozen in a quench using the XXZ model with $\Delta > 1$ even without any disorder [9].

The q-states considered by Ghosh and Žnidarič—at least for $q > 2$ —fall, for the considered small system sizes, also into the category of atypical initial states with small energy fluctuations and an initial dynamics restricted to the boundaries of the finite domain wall. While these states will eventually show extensive fluctuations for fixed q and large system sizes L , a finite-size scaling was not attempted in [1] and the accessible system sizes are likely too small for the larger q-states to ever see typical dynamics.

III. RESONANCES AND QUENCHES FROM A DOMAIN WALL STATE

We have argued above that the domain wall state and the q-states are not suitable to investigate MBL physics because they show only local dynamics at strong disorder and for small system sizes. Here we will provide numerical data to support this point, taking the domain wall state as an example.

The authors of Ref. [1] base their theory of the growth of S_N on resonances between product states connected by the hopping of a single particle. As an example, they show in Fig. 3 of their paper results for quenches from the domain wall state for strong disorder. They argue that in this case, only very few states are involved in the dynamics of the system. We agree with this statement, however, it holds true also for the von Neumann entropy, which unfortunately was not shown in [1]. Both quantities are nearly indistinguishable for the parameters chosen and saturate at short times. This is particularly obvious if one considers the configurational entropy $S_C = S - S_N$, see Fig.1. S_c is extremely small for typical disorder configurations. We conclude that for the considered system sizes and time scales, only single particle dynamics is visible. The system is essentially Anderson localized and the entanglement entropy is caused by number fluctuations alone. While the oscillations at long time can indeed be studied using an effective m -state model, the observed behavior is not related to many-body localization.

IV. POWER-LAW SCALING OF THE NUMBER ENTROPY

The main result of Ref. [1] appears to be the prediction that the number entropy increases like a power law at in-

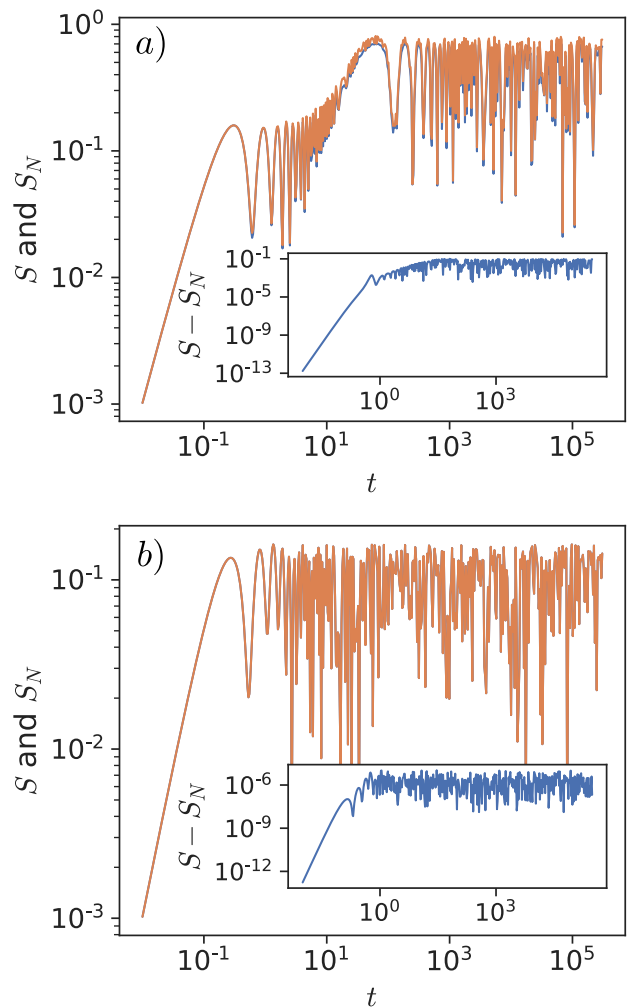


FIG. 1. The analogue of Fig. 3 in [1]. For a quench from the domain wall state and strong disorder $W = 15$, not only the number entropy saturates quickly but also the von-Neumann entropy. (a) Rare disorder configuration: A plateau is visible before the saturation value is reached. (b) Typical disorder configuration: The configurational entropy is extremely small. Note that on this scale S and S_N cannot be distinguished.

intermediate times, followed by a system-size independent saturation. More precisely, the authors write in Eq. (14) of [1] that

$$S_N(t) = \text{const} - B/t \quad (1)$$

with a constant B . Using a quench from the Néel state as an example, we show that this prediction is false. Instead, the growth is well described by

$$S_N(t) = a \ln \ln t \quad (2)$$

with a constant a which does not depend on system size. I.e., the growth is consistent with our earlier results presented in [3–6].

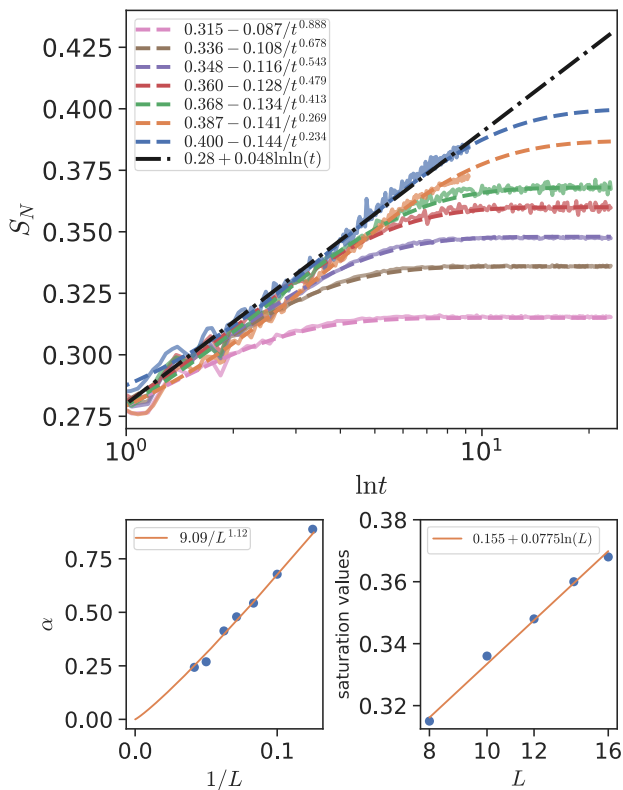


FIG. 2. Quench from the Néel state for $W = 5$ and system sizes $L = 8, 10, \dots, 16, 20, 24$. The left lower panel shows the exponent α of power-law fits, the right lower panel the saturation value as a function of length. We have computed 2×10^5 disorder realizations for $L = 8$ and $L = 10$, 80000 for $L = 12$, 4000 for $L = 14$, 3000 for $L = 16$, 5300 for $L = 20$, and 1406 for $L = 24$. For $L > 16$ we used a Trotter-Suzuki decomposition as described in [3] with $\delta t = 0.035$.

In Fig. 2 we show the number entropy for a quench from the Néel state and a disorder strength $W = 5$. While fits by a power law do work reasonably well close to the finite-size saturation values, the power-law exponent α does depend on system size L and appears to approach zero for $L \rightarrow \infty$. Furthermore, the saturation value is not constant but rather increases $\sim \ln L$. On the other hand, all data fall onto a single $\ln \ln t$ curve before saturation due to the finite size of the systems sets in.

In addition, we also show data for the Hartley entropy at larger disorder strength $W = 8$ in Fig. 3. The Hartley entropy is defined in Ref. [4] and more easily allows to extract the scaling behaviour of the particle fluctuations for very strongly disordered systems. The simple reason is that the Hartley entropy is larger than the number entropy so a proper scaling can already be extracted using a much smaller amount of samples than what would be needed for the number entropy. As for the number en-

tropy, we find that the effective power-law exponent α is not equal to one as predicted by the theory in Ref. [1] but rather tends to zero with increasing system size. The

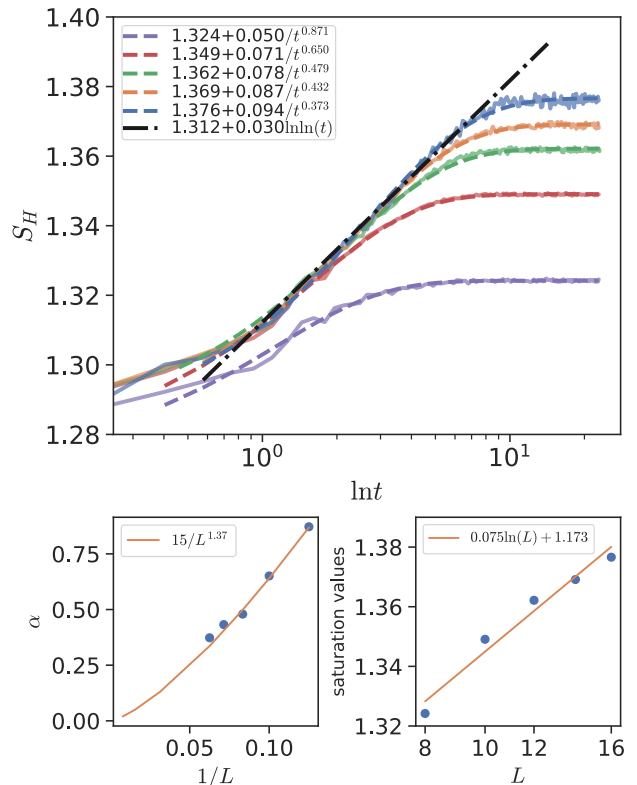


FIG. 3. Same as in Fig. 2 but for the Hartley entropy S_H and for larger disorder $W = 8$. We have computed 10^5 disorder realizations for $L = 8$ and $L = 10$, 20000 for $L = 12$, 10000 for $L = 14$, 2000 for $L = 16$.

saturation values of S_H are also not constant but again increase logarithmically with system size.

To summarize, the theory (1) does not describe the data. They are instead well described by (2).

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