We discuss an open-system analogue of a topological Thouless pump in the steady state of a one-dimensional spin lattice driven by Markovian reservoirs. Periodic variations in a two-dimensional parameter space are shown to lead to quantized transport through the lattice. On an extremal path in parameter space the steady state is pure and coincides with the ground state of the Rice-Mele Hamiltonian for fermions at half filling. Here the quantized transport can be associated with a topological invariant, the Chern number, based on geometric phases. On a more general path, where the steady state is mixed, we introduce the winding of the many-body polarization as a generalized topological invariant. We show that it applies to open quantum systems including effects of interactions and moderate particle-number fluctuations. The winding of polarization, which characterizes the Thouless pump is robust against Hamiltonian perturbations as well as dephasing and particle losses.

Introduction. — Since the discovery of the quantum Hall effect [1] topological states of matter have fascinated scientists in all fields of physics. Topological phases can be characterized by non-local integer invariants [2], whose existence leads to a number of fascinating properties including protected edge states and anyonic excitations [3]. Their robustness against disorder has made topological systems important tools in metrology and promising candidates for robust quantum information platforms [4]. However, topological protection is typically destroyed by dissipation, e.g. decoherence or particle losses. It would be highly desirable to extend the notion of topological order to open systems subject to reservoir interactions. This could substantially extend topological protection since steady states are attractors of the dynamics and thus have an intrinsic robustness.

By now there is a rather good understanding of topological order [5] in non-interacting closed systems. Here an exhaustive classification of gapped topological states (the “ten-fold way” [6]) has been given solely on the basis of general symmetry properties. In contrast, the understanding of topology in interacting systems is limited and the extension to open systems is entirely in its infancy. Very recently there have been some attempts to generalize the concept of topological order to steady states of open systems [7, 8]. This includes proposals to extend Berry phases [9, 10] to mixed states [12-15] using the Uhlmann phase [16], whose suitability has however been questioned [17]. Alternatives suggested for one-dimensional systems are based on projective symmetry group representations [18] but a general tool for the identification of topology in open systems is still missing.

For Gaussian density matrices describing steady states of non-interacting fermionic systems with linear reservoir couplings, where all information is encoded in single-particle correlations, Bardyn et al. developed a classification of topological order [19] employing a scheme similar to the unitary case [5].

Here we pursue a different approach which is applicable also to interacting open systems with non-Gaussian, steady states. Our starting point is an equivalent formulation of geometric Berry (or Zak) phases [9, 10], based on the theory of polarization [11, 20, 21]. We use its many-body formulation [22] and generalize it to open quantum systems. Following the seminal works of Thouless et al. [2, 23], we study cyclic variations in a two-dimensional parameter space of a Liouvillian and show that the winding of the many-body polarization defines a quantized topological invariant, similar to the Chern number.

Specifically we consider a spin chain coupled to Markovian reservoirs described by two parameters. The system

![FIG. 1. (Color online) (a) Open-system analogue of the Rice-Mele model: One-dimensional spin chain with alternating pairwise coupling to two different Markovian reservoirs (Lindblad generators \( L^A_j \) and \( L^B_j \)). The unit cell labelled by \( "j" \) contains a left \( "L" \)(blue) and a right \( "R" \)(green) lattice site. (b) Winding of the many-body polarization along the parameter path where the steady state is a pure state and the ground state of the Rice-Mele Hamiltonian.](image-url)
is a driven-dissipative analogue of the Rice-Mele (RM) model \[24\] and has particle-hole symmetry (PHS). In limiting cases the steady state is pure and equivalent to the ground state of the RM Hamiltonian, which shows a quantized winding of the many-body polarization, corresponding to a quantized bulk transport (Thouless pump). Using numerical simulations and exact analytical arguments we show that winding of the polarization remains strictly quantized also in parameter regions where the steady state is mixed and correlations are present. This winding of the polarization is robust against Hamiltonian perturbations as well as to dephasing and losses, revealing a new regime of topological protection.

Similar to the unitary case \[25\], the polarization is strictly quantized to two possible values differing by one half when the system is inversion symmetric. This generalizes the notion of symmetry protected topological order in open systems.

**Model.**— We consider a translational invariant chain of spin 1/2 particles as shown in Fig. 1(a). Each spin is coupled to two different Markovian reservoirs described by the Lindblad operators

\[
L_j^A = \sqrt{\Gamma(1+\varepsilon)} \left[ (1-\lambda) \left( \hat{\sigma}_{L,j}^+ + \hat{\sigma}_{R,j}^- \right) + \right. \\
+ (1+\lambda) \left( \hat{\sigma}_{L,j}^- + \hat{\sigma}_{R,j}^+ \right) \right], \\
L_j^B = \sqrt{\Gamma(1-\varepsilon)} \left[ (1-\lambda) \left( \hat{\sigma}_{L,j+1}^- + \hat{\sigma}_{R,j}^+ \right) + \right. \\
+ (1+\lambda) \left( \hat{\sigma}_{L,j+1}^+ + \hat{\sigma}_{R,j}^- \right) \right].
\]

(1)

Each unit cell is defined and numbered by the index ”j” and consists of two sites ”R” and ”L”. \(\Gamma\) is the coupling strength to the reservoirs and determines the overall time scale. Our model is characterized by the two parameters \(\lambda \in [-1,1]\) and \(\varepsilon \in [-1,1]\), where \(\varepsilon\) describes the relative strength of reservoir couplings across inequivalent links in the lattice and \(\lambda\) controls the distribution of spin excitations inside a unit cell. For \(\lambda = 1\) both Lindblad operators localize spin excitations on the L-sites. Vice versa for \(\lambda = -1\) the reservoirs drive spin excitations to the R-sites.

We study the steady state \(\rho_{ss}\) of the systems density matrix, which obeys the Master equation in Lindblad form \((\hbar = 1)\) \(\partial_t \rho_{ss} = \mathcal{L} \rho_{ss} = 0\), with

\[
\mathcal{L} \rho = -i[H,\rho] + \frac{1}{2} \sum_{\mu} \{2L_{\mu} \rho L_{\mu}^\dagger - L_{\mu}^\dagger L_{\mu} \rho - \rho L_{\mu}^\dagger L_{\mu} \}
\]

(3)

where \(L_{\mu} \in \{L_j^A, L_j^B\}\). We consider periodic boundary conditions with \(j = 1, \ldots, N\). Because we are interested in reservoir-induced dynamics we set the Hamiltonian \(H\) identical to zero, although later the influence of Hamiltonian perturbations will be discussed.

The following analysis relies on certain symmetries of the Liouvillian, which is invariant under exchange of spin-up and spin-down states and simultaneous spatial inversion with respect to any lattice site. As a consequence one finds that the average number of spin excitations per unit cell is unity, \(\langle \hat{n}_{L,j} + \hat{n}_{R,j} \rangle = 1\), where \(\hat{n} = (\hat{\sigma}_z + 1)/2\), corresponding to half filling with spin excitations or PHS. For our discussion of symmetry protected topological order we furthermore note that the system is inversion symmetric (IS) for \(\varepsilon = 0\) or \(\lambda = 0\).

**Many-body polarization.**— The many-body polarization introduced by Resta \[22\] has been used to describe quantized topological transport and symmetry protected topological order in one dimension on an equal footing. We suggest to generalize eq. \[4\] and apply it to open quantum systems with correlations and weak number fluctuations. Specifically we require that \(\Delta N/N \to 0\) for all relevant states when \(N \to \infty\), which is the case for our system. Here \(\hat{N} = \sum_{\mu} \hat{n}_\mu\) is the total number of excitations and due to PHS we have \(\langle \hat{N} \rangle = N\). \(P\) is defined mod 1, making it invariant under trivial changes of the spin coordinates by \(N\) unit cells. When each spin is shifted by one unit cell the polarization also changes by one unit in the limit \(N \to \infty\). This can be derived from eq. \[4\] using the condition of weak number fluctuations stated above.

Now we use the many-body polarization to study quantized transport in driven-dissipative spin chains introduced by \[1\] and \[2\]. We note that \(P\) contains genuine many-particle contributions and is in general not determined by single-particle correlations alone.

**Pure steady states.**— Let us first consider the steady state for parameters along the special path in parameter space indicated in Fig. 1(b), i.e. for (i) \(\varepsilon = 1\), \(\lambda = 1 \to -1\), (ii) \(\lambda = -1\), \(\varepsilon = 1 \to -1\), (iii) \(\varepsilon = -1\), \(\lambda = -1 \to +1\), and finally (iv) \(\lambda = +1\), \(\varepsilon = -1 \to +1\). Here a pure steady state \(\rho_{ss} = |\Psi\rangle \langle \Psi|\) exists. On (i) and (ii) it factorizes in unit cells \(|\Psi(\lambda)\rangle = \prod_j |\phi(\lambda)\rangle_j\), where up to normalization

\[
|\phi(\lambda)\rangle_j = (1-\lambda) |\uparrow_{L,j} \downarrow_{R,j}\rangle - (1+\lambda) |\downarrow_{L,j} \uparrow_{R,j}\rangle.
\]

In particular for \(\lambda = \pm 1\), \(|\Psi(\lambda)\rangle\) is the Neel state. On parts (iii) and (iv) the steady state factorizes again in new unit cells that are shifted by one site with respect to the old ones. For some parameters the steady state is degenerate. This degeneracy can however easily be lifted by additional terms as will be discussed below. For the dark state one can calculate the polarization analytically,

\[
P = \mp \frac{1}{2} \left( \frac{3}{2} + \frac{\lambda^2}{1 + \lambda^2} \right),
\]

(5)

where the minus sign applies for paths (i,ii) and the plus sign for (iii,iv). As a result \(P\) winds by exactly one unit
when moving along the path discussed above, see Fig. 1b).

Because $|\Psi\rangle$ is a dark state, it is the ground state of the parent Hamiltonian $H_{\text{par}} = \sum_{\mu} J_{\mu}^1 L_{\mu} \geq 0$ which encodes all properties of the steady state. For our model with periodic boundary conditions the parent Hamiltonian reads $H_{\text{par}} = H_{\text{RM}} + H_1$, where

$$H_{\text{RM}} = \sum_j \left( t_A \hat{\sigma}_{L,j}^{+} \hat{\sigma}_{R,j}^{+} + t_B \hat{\sigma}_{L,j+1}^{+} \hat{\sigma}_{R,j}^{+} + h.a. \right) + \Delta \sum_j \left( \hat{\sigma}_{L,j}^{+} \hat{\sigma}_{L,j} - \hat{\sigma}_{R,j}^{+} \hat{\sigma}_{R,j} \right) + 4\Gamma(1 + \lambda^2) \quad (6)$$

is the spin version of the Rice-Mele Hamiltonian. The parameters are given by $t_A = 2\Gamma(1 + \varepsilon)(1 - \lambda^2)$, $t_B = 2\Gamma(1 - \varepsilon)(1 - \lambda^2)$ and $\Delta = 8\Lambda\Gamma$. This model is a paradigmatic example for a topological Thouless pump [23]. When following the systems ground state adiabatically while $t_A - t_B$ and $\Delta$ encircle the topological singularity at $t_A = t_B$, $\Delta = 0$, a quantized current is induced in the bulk. It can be associated with a quantized winding of the many-body polarization, corresponding to an integer topological Chern number ($C = 1$). The second term $H_1$ has no effect on the dark state since $H_1 |\Psi(\lambda)\rangle = 0$.

**Mixed steady states.** — In the following we will analyze the polarization when changing parameters of the Liouvillian on a more general path, where the steady-state is mixed. In addition, to lift the degeneracy mentioned above Eq.(5), we introduce a homogeneous magnetic, $H = \nu \sum_j (\hat{\sigma}_{L,j}^{z} + \hat{\sigma}_{R,j}^{z})$ and set $\nu = \Gamma$ for convenience. Moreover we add small but generic non-linear terms to the Lindblad generators

$$L_j^{A} \rightarrow L_j^{A} + \sqrt{\Gamma(1 + \varepsilon)} \left( \hat{\sigma}_{L,j}^{+} \hat{\sigma}_{R,j}^{+} - \hat{\sigma}_{L,j}^{-} \hat{\sigma}_{R,j}^{-} \right), \quad (7)$$

$$L_j^{B} \rightarrow L_j^{B} + \sqrt{\Gamma(1 - \varepsilon)} \left( \hat{\sigma}_{R,j}^{+} \hat{\sigma}_{L,j+1}^{+} - \hat{\sigma}_{R,j}^{-} \hat{\sigma}_{L,j+1}^{-} \right). \quad (8)$$

They prevent the steady state from becoming completely mixed on the $\lambda = 0$ axis (see supplementary material for details). These additional terms do not change the dark state on the extremal path nor do they affect the inversion symmetry on the $\varepsilon = 0$ and $\lambda = 0$ axis.

We calculate $P$ numerically for four sites with periodic boundary conditions or using the time-dependent block decimation (TEBD) algorithm [25] for larger systems (see supplementary material for details). In Fig. 2 we plot the polarization in the whole parameter space. One recognizes a strictly quantized winding when encircling the origin.

This result is rather surprising at first glance. Although the steady state is no longer pure and does not commute with the total particle number, there is a strictly quantized winding of the many-body polarization. This winding has a clear interpretation: During a full cycle around the origin $\varepsilon = \lambda = 0$ one spin excitation per lattice site is moved by one lattice constant on average. The quantized winding of the polarization defines a topological invariant which characterizes families of non-Gaussian steady states on paths through parameter space. On the extremal path (see Fig. 1) the interpretation as particle transport is evident from the connection to the RM model. In the general case this is less clear since there is no conserved current flowing into and out of the reservoirs. However, as will be shown in [27], the winding of the polarization can be connected to a true quantized particle transport induced in a second system with conserved particle number coupled to the open spin chain.

**Symmetry-protected topological order.** — Next we want to study systems with additional symmetries, focusing concretely on spatial inversion. Let us consider symmetric systems with polarizations $P_1$ and $P_2$ respectively. Because under inversion the difference of their polarizations $\Delta P = P_1 - P_2$ changes its sign $\Delta P \rightarrow -\Delta P$, but at the same time has to remain invariant because of IS, it follows that

$$\Delta P = -\Delta P \mod 1. \quad (9)$$

Therefore the difference $\Delta P$ can only take two quantized values $\Delta P = 0, 1/2$ [10].

For IS systems at $\lambda = 0$, $\varepsilon = \pm 1$ and $\varepsilon = 0$, $\lambda = \pm 1$ where the steady state is pure, we obtain two pairs of symmetry protected topological phases of the RM model with $\Delta P = 1/2$ respectively. For parameters where the steady state is mixed the quantization of the polarization remains intact as long as the system is IS, see Fig. 2. Therefore a topological transition takes place in the center $\lambda = \varepsilon = 0$, where $P$ is not defined and $\Delta P$ can change discontinuously. Indeed, from symmetries we can show in general that $|P_{\lambda=0}(\varepsilon) - P_{\varepsilon=0}(\lambda)| = 1/2$ and $|P_{\lambda=0}(\varepsilon) - P_{\lambda=0}(\varepsilon)| = 1/2$. 

**FIG. 2.** (Color online) Many-body polarization of the steady state calculated from exact simulation of a four-site problem with periodic boundary conditions. $P$ shows a quantized winding by one unit upon a full cycle around the origin. We have verified this for larger systems using TEBD simulations of the full density matrix. One notices the quantization of the polarization along the two inversion symmetric axes $\varepsilon = 0$ and $\lambda = 0$ with a topological phase transition at the origin.
We conclude that the many-body polarization can be used to generalize the notion of symmetry protected topological order to open systems with correlations and moderate particle-number fluctuations. We expect that other symmetries besides spatial inversion can also be treated in this way.

Robustness.— Closed systems with topological order are robust against perturbations that do not close the energy gap. This makes it interesting to ask whether the winding of the polarization in the open system is robust against Hamiltonian and Liouvillian perturbations. To investigate robustness in sufficiently large systems \((N = 16)\) we evolved the stationary state along a circular path as indicated in Fig. 3(a). We analyzed three different types of perturbations: (i) a spatially random magnetic field in the \(z\)-direction, (ii) dephasing, and (iii) local dephasing with homogeneous losses and pumping both with equal rates. In the first case we add a Hamiltonian \(H_{\text{pert}} = \sum_{\mu} \gamma_\mu \sigma_\mu^z \), where \(\mu\) denotes a lattice site and \(\gamma_\mu \in [-1,1]\) are independent random magnetic field strengths. As can be seen from Fig. 3(b) this perturbation has negligible influence on the polarization and does not affect its winding. In the second case we add an additional Lindblad term with \(L_{\text{deph}}^{R/L,j} = \sqrt{\gamma_\perp} \sigma_j^z\), where \(\gamma_\perp\) is the dephasing rate. This term, too, has almost no effect on \(P\) and does not change its winding. The same is true for homogeneous losses and gain. These have to be added with equal rates \(\gamma_\parallel\) however in order to maintain the PHS of the system, i.e. \(\langle \hat{n}_L + \hat{n}_R \rangle = 1\): \(L_{\text{deph}}^{R/L,j} = \sqrt{\gamma_\parallel} \sigma_j^+\), and \(L_{\text{deph}}^{R/L,j} = \sqrt{\gamma_\parallel} \sigma_j^-\). The small effects of these non-hermitian perturbations are shown in Fig. 3(c). This finding is important, because it shows that in an open system topological order and the associated quantized charge pump can be maintained even in the presence of unwanted losses, if the system remains PHS.

Topological singularity.— For open Gaussian systems it has been argued in \([19]\) that a topological singularity, at which a topological phase transitions takes place, is connected to the closing of at least one of the two, the damping gap or the so-called purity gap. The damping gap is determined by the real part of the eigenvalues of \(\mathcal{L}\), \(\mathcal{L}_{\text{pert}} = \epsilon \mathcal{P}_c\), which characterize the relaxation rate of the system. The steady state is a right eigenstate of \(\mathcal{L}\) with eigenvalue \(\epsilon = 0\) and the damping gap \(\Delta_\text{d}\) is the distance to the next larger value of \(\text{Re}[\epsilon]\). The purity gap of fermionic systems is determined by the eigenvalue spectrum of the positive semidefinite matrix \((i\gamma)^2\), where \(\gamma\) is the antisymmetric and real steady state single-particle correlation matrix \([19]\).

The damping gap is difficult to obtain from TEBD simulations and we have performed exact simulations for a four-site problem. While even for such small system sizes we can already observe a quantized winding of the polarization, the damping gap is sizable everywhere. Since our spin system cannot be mapped to a free fermionic model the concept of the purity gap introduced in \([19]\) cannot be applied. However for small systems with four sites we can evaluate the eigenvalue spectrum of the full density matrix. We find that the two dominant eigenvalues of \(\rho_{\text{loc}}\) never become equal (see supplementary for details). Although these calculations can only be done for finite systems this indicates that there is no closing of a “generalized” purity gap either, since an indicator of a topological transition is expected to be independent of system size. A necessary condition for a singularity in the polarization is the vanishing of the expectation value in Eq. (4). The physical interpretation of this is however unclear and thus conditions for a topological singularity and its nature in a dissipative system remain open questions devoted to further studies.

Summary.— To summarize, we have generalized the notion of topology to steady states of one-dimensional interacting open systems. We investigated an open spin chain with reservoir couplings that lead to particle-hole symmetric steady states. We defined a topological invariant by the quantized winding of the many-body polarization which characterizes Thouless pump. While topological invariants such as the Zak phase or Chern number for closed systems, or invariants based on the single-particle density matrix, are no longer applicable, the quantized bulk transport of the Thouless pump remains a suitable indicator of topology in more general one-dimensional open systems. In the presence of inversion symmetry distinct phases with symmetry protected topological order can be defined which can be distinguished by the value of the many-body polarization. The polarization is a measurable quantity even in an open system and specific detection protocols will be discussed in detail elsewhere \([27]\). We have shown that the polarization winding is ro-

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{(Color online) Many-body polarization under slow variation of the parameters evaluated along a circular path (red) as indicated in (a) with radius \(R = \sqrt{x^2 + y^2} = 1\) using density-matrix TEBD simulations in the presence of different perturbations \((N = 16\) unit cells): (b) spatially random magnetic field in \(z\) direction, \(V/\Gamma = 1\) and 2 (dashed) and (c) local dephasing with homogeneous losses and pumping both with rates \(\gamma_\perp/\Gamma = 0.5\) and 1(dashed), where \(\gamma_\parallel = \gamma_\perp\).}
\end{figure}
bust to Hamiltonian disorder as well as to additional dephasing or moderate particle losses. The conditions for a topological singularity remain an open question which requires further investigation. An interesting future avenue is the extension to higher spatial dimensions which may provide a way to realize topologically protected transport in an open system that is robust against particle losses.

Acknowledgment.— We acknowledge useful discussions with S. Diehl, S. Huber, E.v.Nieuwenburg and M. Koster. FG acknowledges financial support from the Gordon and Betty Moore foundation.

[27] D. Linzner and M. Fleischhauer (in preparation)

SUPPLEMENTARY

Numerical method.— Aside from the extremal path, as seen in Fig.[1] efficient numerical methods are needed to get insight into the dynamics of large systems. For this purpose we utilize an extension of the time-evolving-block-decimation algorithm (TEBD)[26, 28].

This method was originally conceived for efficient unitary time evolution of pure states in closed system and relies on the controlled truncation of the underlying Hilbert space, which is feasible for all moderately entangled states. Using small modifications however, the TEBD-algorithm can be easily extended to time evolution of mixed states in open systems. For this we embed the mixed state as a pure state in a larger Hilbert space, the so-called Liouville space using the isomorphism

\[ \rho = \sum_{\alpha,\beta} \rho_{\alpha,\beta} |\alpha\rangle\langle\beta| \leftrightarrow \sum_{\alpha,\beta} \rho_{\alpha,\beta} |\alpha\rangle|\beta\rangle \equiv |\rho\rangle, \]

The state now evolves under the Schrödinger-like equation

\[ \partial_t |\rho\rangle = \mathcal{L}|\rho\rangle, \]

with \( \mathcal{L} \) being the representation of \( \mathcal{L} \) in Liouville space.
Our results are obtained in large systems (N=16) with open boundary conditions. To minimize finite size effects Eq. (4) is evaluated within a region inside the bulk of the system (N=14) resulting in a small error because per definition periodic boundaries are required and correlations between the edges are neglected. However this error is suppressed with increasing system size and vanishes in the thermodynamic limit as in our system spin-spin correlations decay exponentially on a length scale of one lattice site.

Necessity of non-linear modification. — On the whole $\lambda = 0$-axis Lindblad generators (1) and (2) become hermitian. Thus $[L_\mu, L_\mu^\dagger] = 0$ holds and the completely mixed state

$$\rho_M = \bigotimes_j \frac{1}{2} ( |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| )$$

is a steady state of the system. In fact one can easily show that for this state Eq. (4) is not defined as the argument of the logarithm becomes zero. To prevent a completely mixed steady state on the $\lambda = 0$ axis we add non-linear terms (7) and (8) to the original model.

Damping and purity gap. — Although not attainable by TEBD, we can calculate the exact damping and purity spectrum of a small four-site spin chain, as can be seen in Fig.4 and Fig.5 respectively. In order to consider a most general case, while a quantized winding of the polarization is still observable, we include an additional dephasing. As can be seen from Fig.4 there is always a finite damping gap. Furthermore as can be seen from Fig.5 the dominant eigenvalues of the density matrix do not become degenerate.

FIG. 4. (Color online) Two lowest eigenvalues of $L$ as function of reservoir parameters $\lambda$ and $\varepsilon$ with dephasing $\gamma_\perp = 0.5\Gamma$. One finds no closure of the damping spectrum.

FIG. 5. (Color online) Spectrum $\sigma(\rho_{ss})$ of the steady-state density matrix for additional homogeneous dephasing $\gamma_\perp = 0.5\Gamma$ at $\varepsilon = 0$ obtained from exact diagonalization of a four-site problem as function of $\varepsilon$ for $\lambda = 0$ (a), and as function of $\lambda$ with $\varepsilon = 0$. 