chiralities (29), which clearly confirms this quantization within the experimental errors. Our measurements demonstrate that even when the global topology has Chern number zero, the distribution of Berry curvature can be very rich.

Our measurement scheme can be readily extended to characterize bands with Chern numbers differing from zero (47, 19). In principle, one could start in a shallow lattice, where reaching nonzero Chern numbers is feasible, and for the tomography project onto flat bands, which can be reached, such as by dynamical control over the offset. Our method for generating the topological bands is spin-independent and does not couple different spin states. It therefore can be extended to high-spin systems (29) or to strongly interacting spin mixtures, which are expected to lead to interesting many-body phases (30–32).

REFERENCES AND NOTES
28. Materials and methods are available as supplementary materials on Science Online.

ACKNOWLEDGMENTS
We acknowledge stimulating discussions with A. Eckardt, M. Lewenstein, L. Mathey, and C. Strüter. This work has been supported by the excellence cluster "The Hamburg Centre for Ultrafast Imaging—Structure, Dynamics, and Control of Matter at the Atomic Scale" and the GZ 1356 of the Deutsche Forschungsgemeinschaft.

SUPPLEMENTARY MATERIALS
www.sciencemag.org/content/352/6289/1094/suppl/DC1
Supplementary Text
Figs. S1 and S2
References (33–36)
18 September 2015; accepted 23 March 2016
10.1126/science.aad4568

QUANTUM SIMULATION

Bloch state tomography using Wilson lines
Tracy Li, Lucia Duca, Martin Reitter, Fabian Grusdt, Eugene Demler, Manuel Endres, Monika Schlierer-Smith, Immanuel Bloch, Ulrich Schneider

Topology and geometry are essential to our understanding of modern physics, underlying many foundational concepts from high-energy theories, quantum information, and condensed-matter physics. In condensed-matter systems, a wide range of phenomena stem from the geometric properties of band eigenstates, which is encoded in the matrix-valued Wilson line for general multiband systems. Using an ultracold gas of rubidium atoms loaded in a honeycomb optical lattice, we realize strong-force dynamics in Bloch bands that are described by Wilson lines and observe an evolution in the band populations that directly reveals the band geometry. Our technique enables a full determination of band eigenstates, Berry curvature, and topological invariants, including single- and multiband Chern and Z_2 numbers.

GEOMETRIC TOPOLOGY

Using ultracold atoms in a graphene-like honeycomb lattice, we demonstrate that Wilson lines can be accessed and used as versatile probes of band structure geometry. Whereas the Berry phase merely multiplexes a state by a phase factor, the Wilson line is a matrix-valued operator that can mix state populations (II) (Fig. 1A). We measure the Wilson line by detecting changes in the band populations (17) under the influence of an external force, which transports atoms through reciprocal space (18). In the presence of a force F, atoms with initial quasimomentum q(0) evolve to quasimomentum q(t) = q(0) + Ft/h after a time t. If the force is sufficiently weak and the bands are nondegenerate, the system will undergo adiabatic Bloch oscillations and remain in the lowest band (18). In this case, the quantum state merely acquires a phase factor composed of the geometric Berry phase and a dynamical phase. At stronger forces, however, transitions to other bands occur, and the state evolves into a superposition of several bands.

When the force is infinite with respect to a chosen set of bands, the effect of the dispersion vanishes, and the bands appear as effectively degenerate (Fig. 1B). The system then evolves according to the formalism of Wilczek and Zee for adiabatic motion in a degenerate system (II). The unitary time-evolution operator describing the dynamics is the Wilson line matrix (19)

\[ W_{q(0)\rightarrow q(t)} = \exp\left( i \int dq \mathbf{A} \right) \]

where the path-ordered (P) integral runs over the path C in reciprocal space from q(0) to q(t) and \( \mathbf{A} \) is the Wilczek-Zee connection, which encodes the local geometric properties of the state space. In a lattice system with Bloch states \( \Phi_{\mathbf{k}_{n}}(x) = e^{i\mathbf{k}_{n}\cdot \mathbf{r}} |\mathbf{k}_{n}\rangle \) in the nth band at quasimomentum \( \mathbf{q} \), where \( \mathbf{r} \) is the position operator, the elements of the Wilczek-Zee connection are determined by the cell-periodic part \( |\mathbf{k}_{n}\rangle \) as \( \mathbf{A}_{n} = i \langle \mathbf{k}_{n} | \mathbf{W}_{\mathbf{q}(\mathbf{r})} |\mathbf{k}_{n}\rangle \). The diagonal elements \( (n = n) \) are the Berry connections of the individual Bloch bands, which yield the Berry phase when integrated along a...
closures. The off-diagonal elements \( (n \neq n') \) are the interband Berry connections, which couple the bands and induce interband transitions.

Although the evolution described by Eq. 1 must be path-ordered when the Wilczek-Zee connections at different quasi-momenta do not commute, it can also be path-independent under certain circumstances \((9, 20)\). For example, when the relevant bands span the same Hilbert space for all quasi-momenta, as is the case in our system, the Wilson line operator describing transport of a Bloch state from \( Q \) to \( q \) reduces to \( W_{Q \rightarrow q} = e^{i \theta Q \rightarrow q} \) \((9, 19, 21)\). Consequently, the elements of the Wilson line operator simply measure the overlap between the cell-periodic Bloch functions at the initial and final quasi-momenta \( (9, 21) \).

\[
W_{Q \rightarrow q} = \langle \phi_q^{\text{in}} | e^{i \theta Q \rightarrow q} | \phi_Q^{\text{in}} \rangle = \langle u_Q | u_q \rangle
\]

Hence, access to the Wilson line elements facilitates the characterization of band structure topology in both path-dependent and path-independent evolution. In both cases, the topological information is encoded in the eigenvalues of the Wilson-Zak loops. In the latter case, the simplified form of the Wilson line in Eq. 2 additionally enables a map of the cell-periodic Bloch functions over the entire Brillouin zone (BZ) in the basis of the states \( | \Phi_Q \rangle \) at the initial quasi-momentum \( Q \).

We create the honeycomb lattice by interfering three blue-detuned laser beams at 120(1)° angles (Fig. 2A). At a lattice depth \( V_0 = 5.2(1)E_r \), where \( E_r = \hbar^2/(2m^2 r) \) is the recoil energy, \( \lambda_q \) is the laser wavelength, and \( m \) is the mass of \(^{87}\text{Rb} \), the combined width \( \varepsilon = \hbar \times 3 \) kHz of the lowest two bands is much smaller than the \( \hbar \times 15 \) kHz gap to higher bands. Consequently, there exists a regime of forces where transitions to higher bands are suppressed, and the system is well approximated by a two-band model \((9)\).

We probe the lattice geometry with a nearly pure Bose-Einstein condensate of \(^{87}\text{Rb} \), which is initially loaded into the lowest band at quasi-momentum \( q = \Gamma \), the center of the BZ (Fig. 2B). To move the atoms in reciprocal space, we linearly sweep the frequency of the beams to uniformly accelerate the lattice, thereby generating a constant inertial force in the lattice frame. By independently controlling the frequency sweep rate of two beams (Fig. 2A), we can tune the magnitude and direction of the force and move the atoms along arbitrary paths in reciprocal space.

To verify that we can access the Wilson line regime, where the dynamics are governed entirely by geometric effects, we transport the atoms from \( \Gamma \) to different final quasi-momenta using a variable force \( |F| \) and perform band-mapping \((17)\) to measure the population remaining in the lowest band (Fig. 2C). For vanishing forces, we recover the adiabatic limit, where the population remains in the lowest band. For increasing forces (i and ii in Fig. 1B), where \( |F|/d > \varepsilon \), the population continues to decrease. However, at strong forces (iii in Fig. 1B), where \( |F|/d > \varepsilon \), the population saturates at a finite value. For example, after transport by one reciprocal lattice vector (blue data in Fig. 2C), about one-quarter of the atoms remain in the first band, in stark contrast to typical Landau-Zener dynamics, where the population vanishes for strong forces \((22)\).

![Fig. 1. Wilson lines and effectively degenerate Bloch bands. (A) In a non-degenerate system (left), adiabatic evolution of a state through parameter space \( \mathbf{R} \) results in the acquisition of a geometric phase factor, known as the Berry phase. In a degenerate system (right), the evolution is instead governed by a matrix-valued quantity called the Wilson line. If the degenerate levels can be experimentally distinguished (blue and yellow coloring), then population changes between the levels are detectable. (B) The band structure of the lowest two bands of the honeycomb lattice is given in effective energy units of \( |F|/d \), where \( F \) is the applied force used to transport the atoms and \( d \) is the distance between nearest-neighbor lattice sites. As \( |F| \) is increased, the largest energy scale of the bands becomes small compared to \( |F|/d \). At large forces (ii), the effect of the band energies is negligible, and the system is effectively degenerate. In this regime, the evolution is governed by the Wilson line operator. We distinguish between the bands using a band-mapping technique that detects changes in the band population along the Wilson line path.](image)

![Fig. 2. Reaching the Wilson line regime in the honeycomb lattice. (A) Schematic of the honeycomb lattice in real space with A (B) sublattice sites denoted by solid (open) circles. The lattice is formed by interfering three in-plane laser beams (blue arrows) with frequency \( \omega \). Sweeping the frequency of beam \( i \) by \( \delta \omega_1 \) creates a force \( F_1 \) in the lattice frame in the propagation direction of beam \( i \) \((9)\). (B) Two copies of the first BZ of the honeycomb lattice, separated by a reciprocal lattice vector \( G \). By changing the relative strengths of \( F_1 \) (red arrows), the atoms can be moved along arbitrary paths in reciprocal space. Each BZ features nonequivalent Dirac points \( K \) and \( K' \) at the corners of the hexagonal cell. High-symmetry points \( \Gamma \), at the center of the BZ, and \( M \), at the edge of the BZ, are also shown. (C) The population remaining in the first band for different forces after transport to \( \Gamma + 0.2G \) (green), \( \Gamma + 0.55G \) (red), and \( \Gamma + G \) (blue). Inset numbers i to iii refer to band schematics in Fig. 1B, representing the diminishing effect of the dispersion for increasing force. The data agree well with a two-level, tight-binding model (dashed line) that approaches the Wilson line regime (thick shaded line) at large forces. Discrepancies at larger forces result from transfer to higher bands and match well with ab initio theory using a full band structure calculation including the first six bands (thin solid line) \((9)\). For all subsequent data, we use \( |F|/d = 4.8 \), indicated by the dashed gray line. Error bars indicate the standard error of the mean (SEM) from 10 shots per data point.](image)
Theoretically, the population in the first band after the strong-force transport directly measures the Wilson line element \( |W_{\Gamma\rightarrow 1}|^2 = |\langle \phi_1^\alpha | \bar{W}_{\Gamma\rightarrow 1} | \phi_1^\beta \rangle |^2 \) in the basis of the band eigenstates. Based on Eq. 2, the saturation value \( |W_{\Gamma\rightarrow 1}|^2 = |\langle \bar{u}_\Gamma^\alpha | \bar{u}_\Gamma^\beta \rangle |^2 \) of the population after transport to \( \Gamma \) is a measure of the overlap between the cell-periodic Bloch functions of the first band \( |\bar{u}_\Gamma^\alpha \rangle \) at \( \Gamma \) and \( \Gamma \). Notably, for the case of transport by one reciprocal lattice vector \( G \), the cell-periodic parts \( |\bar{u}_\Gamma^\alpha \rangle \) are not identical, due to the finite saturation value.

To corroborate that our experiment measures the Wilson line, we transport atoms initially in the ground state at \( \Gamma \) by up to three reciprocal lattice vectors (Fig. 3). The threefold rotational symmetry of the lattice, combined with the symmetry of its s-orbitals, makes the path from \( \Gamma \) to \( \Gamma + 3G \) equivalent to the triangular path shown in Fig. 3A, such that the overlap between cell-periodic components of the Bloch wave functions at the two endpoints is unity (see Eq. 2). Correspondingly, we expect to recover all the population in the lowest band after transport from \( \Gamma \) to \( \Gamma + 3G \). This prediction is confirmed in Fig. 3B, where we plot the population remaining in the first band after transport to final quasimomentum \( q \). The data are well described by a tight-binding model that takes into account the relative phase between orbitals on A and B sites of the lattice due to the Wilson line \( W_{\Gamma\rightarrow 1} = e^{i\theta} \). Physically, this can be understood by assuming that the real-space wave function simply accumulates a position-dependent phase when the strong force \( F = \hbar (q - Q) / t \) is applied for a short time \( t \) (Fig. 3C).

As the Wilson line enables a comparison of the cell-periodic Bloch functions at any two quasimomenta (Eq. 2), it can in principle be applied to fully reconstruct these states throughout reciprocal space. To this end, it is convenient to represent the state \( |\bar{u}_\Gamma^\alpha \rangle \) at quasimomentum \( q \) in the basis of cell-periodic Bloch functions

---

**Fig. 3. Measuring mixing angles \( \theta \) at different quasimomenta.** (A) Because of the threefold-rotational symmetry of the honeycomb lattice, a path from \( \Gamma \) to \( \Gamma + 3G \) is equivalent to a triangle-shaped path with each leg of length \( |G| \), beginning and ending at \( \Gamma \). Colored dots correspond to colored quasimomentum labels in (B). (B) The population remaining in the first band after transport to final quasimomentum \( q \). Theory lines are a single-particle solution to the dynamics using a full lattice potential and including the first six bands (solid) and a two-band, tight-binding model (dashed) (19). The inset Bloch sphere depicts the transported state at \( \Gamma \) (red), \( \Gamma + G \) (blue), and \( \Gamma + 2G \) (green) in the basis of the cell-periodic Bloch functions at \( \Gamma \). Error bars represent the SEM from averaging 9 to 11 shots, with the exception of \( q = M + G \) and \( q = M + 0.9G \), which show the average of 20 shots. (C) Transport of a Bloch state by one reciprocal lattice vector corresponds to a 2\( \pi \) phase shift in the real-space wave functions of each sublattice site. Projecting the combined lattice and gradient potential \( V(x) \) along the path shown in red onto the x axis, which is the direction of the applied force in the measurements of Figs. 2B and 3B, highlights the effect of the real-space embedding of the honeycomb lattice. Since the distance between A (solid circles) and B sites (open circles) is 1/3 the distance between sites of the same type, there is a phase difference of 2\( \pi/3 \) between the real-space wave functions of A and B sites, which gives rise to the band mixing.

---

**Fig. 4. Measuring relative phases \( \phi_\alpha \) at different quasimomenta.** (A) Schematic of the interferometric sequence in the extended BZ scheme (left) and the corresponding rotation on the Bloch sphere (right). To create a superposition state, atoms initially in the lowest eigenstate at \( \Gamma - G \) are rapidly transported to \( \Gamma \). The phase of the superposition state is controlled by varying the hold time at \( \Gamma \) (ii). After the state preparation, the atoms are transported to a final quasimomentum \( q_\alpha \), which is parametrized by the angle \( \alpha \) and lies on a circle of radius |G| centered at \( \Gamma \) (iii). (B) Phases \( \phi_\alpha \), referenced to \( \alpha = 180^\circ \) for the lattice with AB-site degeneracy (blue) and AB-site offset (red). Data in blue have been offset by +120° for visual clarity. Dashed lines are a two-band, tight-binding calculation with \( \Delta/J = 0 \) (blue) and \( \Delta/J = 3.1 \) (red), where \( J = \hbar \times 500(10) \) Hz. Error bars indicate fit errors.
[1] = |\psi_{Q1}|^2 and [2] = |\psi_{Q2}|^2 at a fixed reference quasimomentum Q as

\[ |\psi_{Q1}|^2 = \cos \theta Q(1) + \sin \theta Q e^{i\phi Q}(2) \]  

(3)

Mapping out the geometric structure of the lowest band therefore amounts to obtaining \( \theta Q \) and \( \phi Q \), which parametrize the amplitude and phase of the superposition between the reference Bloch states, for each quasimomentum \( Q \). The total phase of \( |\psi_{Q1}|^2 \) is fixed for all \( Q \) by the total quantum number \( n \) of the band and is thereby determined by the phases \( \psi_{Q1} \) and \( \psi_{Q2} \) of the corresponding cell-periodic functions (inset of Fig. 3B). When we remove this constraint by introducing an energy offset between \( A \) and \( B \) sites (39, 30), we observe smooth varying phases that are always less than \( \pi \) (red circles in Fig. 4B). The dependence of the phase on angle \( \alpha \) indicates both the breaking of inversion symmetry and the preservation of the three-fold rotational symmetry of the lattice. Apart from reconstructing the cell-periodic Bloch functions, our method also provides access to eigenvalues of Wilson-Zak loops, \( \bar{W}_{q \rightarrow -q}^{\Gamma} \) which is essential for determining various topological invariants (7–9). To this end, we split the Wilson-Zak-loop matrix into a global phase factor, which can be measured by extending previous methods (13–15, 30), and an SU(2) matrix with eigenvalues \( e^{i\xi} \). Using the data from Figs. 3B and 4B, we reconstruct the eigenvalues for a loop transporting from \( \Gamma \) to \( G \), up to multiplets of \( \pi \) (19). We find the eigenvalue phases to be \( \xi = 1.03(2) \pi /3 \), in good agreement with the value of \( \pi /3 \) predicted from the two-band model. Notably, we measure the same eigenvalues even when the band eigenstates are modified by an energy offset between \( A \) and \( B \) sites (29). This invariance is a direct consequence of the real-space representation of the Wilson-Zak loop, \( \bar{W}_{q \rightarrow -q}^{\Gamma} = e^{-i\Gamma} \) [see Eq. 2 and (19)]. Because the Wilson-Zak loop depends only on the position operator \( \bar{r} \), the eigenvalues are determined solely by the physical locations of the lattice sites, which are unchanged by the energy offset.

Our versatile approach only employs standard techniques that are broadly applicable in ultracold-atom experiments and can be extended to higher numbers of bands by adopting ideas from quantum process tomography (32). Provided that the relevant local Hilbert space is identical for all quasimomenta, our method provides a complete map of the eigenstates over the BZ, giving access to the Berry curvature and Chern number. When this is not the case, the same techniques enable the reconstruction of Wilson-Zak loop eigenvalues, which directly probe the geometry of the Wannier functions (9 and, therefore, the polarization of the system (21, 33)). Consequently, these eigenvalues can reveal the topology of bands with path-dependent and non-Abelian Wilson lines (9, 11). Such systems can be realized in cold-atom experiments by periodically modulation the lattice (12, 30, 34–36) to create a quasimomentum-dependent admixture of additional bands (37) or coupling between internal states (38–41). Moreover, the addition of spin-orbit coupling (42) would enable the investigation of the \( Z_2 \) invariant characterizing time-reversal invariant topological insulators (33, 41, 43–46).

REFERENCES AND NOTES
24. Acknowledgments
We acknowledge illuminating discussions with A. Alexandradinata, J.-N. Fuchs, N. Goldman, D. Greiter, L.-K. Lim, G. Montabaux, A. Polkovnikov, and G. Refael. This work was supported by the Alfred P. Sloan Foundation, the European Commission (QOAM, AQUS), National Systems Initiative Munich, the Harvard Quantum Optics Center, the Harvard-Massachusetts Institute of Technology Center for Ultracold Atoms, NSF grant DMR-1308435, the Defense Advanced Research Projects Agency Optical Lattice Emitter program, the Air Force Office of Scientific Research, Quantum Simulation Multidisciplinary University Research Initiative (MuRI), the Army Research Office (ARO)–MuRI on Atomtronics, and the ARO-MURI Qubit Enabled Imaging, Sensing, and Metrology program.

SUPPLEMENTARY MATERIALS
www.sciencemag.org/content/352/6289/1097/suppl/DC1
Supplementary Text
Figs. S1 to S7
References (47–52)
5 October 2015; accepted 15 April 2016
10.1126/science.aad5812