Abstract

Photon Bose-Einstein condensates are characterised by a quite weak interaction, so they behave nearly as an ideal Bose gas. Moreover, since the current experiments are conducted in a microcavity, the longitudinal motion is frozen out and the photon gas represents effectively a two-dimensional trapped gas of massive bosons. In this paper we therefore focus on a harmonically confined ideal Bose gas in two dimensions, where the anisotropy of the confinement allows for a dimensional crossover. If the anisotropy is even large enough so that the squeezed direction is frozen out, then only one degree of freedom survives and the system can be considered to be quasi-one dimensional. We work out the thermodynamic properties for such a system analytically and examine, in particular, the dimensional information which is contained in the respective thermodynamic quantities. With this our results are useful for future experiments of photon gases at the dimensional crossover from 2D to 1D in view of determining their effective dimensionality from thermodynamic quantities.

1 Introduction

The question of Bose-Einstein condensation in lower dimensions got already tackled quite early in the post-war era of physics. Soon it was found out that in the case of lower dimensional systems without trapping potential no long-range order can emerge and, thus, no Bose-Einstein condensation in such systems is possible. Later on in the early 1990s but prior to the experimental realisation of Bose-Einstein condensates (BEC), the authors of Ref. worked out that, with the aid of an external trapping potential, the excited states of lower-dimensional ideal Bose gases can saturate, meaning that Bose-Einstein condensation is possible. In the thermodynamic limit they showed for a trapping potential, which is stronger confining than a box, that a condensation in 2D can occur, whilst in a 1D setting a potential more confining than a quadratic potential is necessary. After the experimental realisation of BECs naturally the question came up, how to achieve systems with an effective dimension lower than three. In Ref. the question of the effective dimensionality of the system is reduced to a comparison of different length scales, which are in a 3D axially symmetrically trapped Bose gas the in-plane radius, the axial width, the scattering length, and the healing length. A system is effectively 2D, if the healing length is larger than the axial width, and effectively
1D, if the healing length is larger than the in-plane radius with the axial width being still larger than the healing length. As the healing length is inversely proportional to the square root of both the density and the scattering length, one can control the effective dimension either by changing the density, as is done in Ref. [6], or by changing the interaction strength itself via a Feshbach resonance [7].

Another possibility of tuning the effective dimension of a system is to modify the kinetic energy in a certain direction. This can be achieved in lattice systems by changing the hopping matrix amplitude. One possible experimental realisation is to use coupled 2D BECs in order to perform a crossover from 2D to 3D [8]. A more recent experiment consists of 2D arrays of coupled 1D Bose gases [9][11]. Decreasing the 2D lattice depth yields an increase of the hopping amplitude and thus gives rise to a dimensional crossover to higher dimensions.

In the following, however, we dedicate our discussion for the sake of concreteness on photon BECs as realised in Ref. [12][14]. Due to the experimental setup, these systems are intrinsically two-dimensional and almost non-interacting. Thus, the question remains how to manipulate the system to be effectively one-dimensional. The aim of this paper is to show, how this can be achieved by using highly anisotropic harmonic trapping potentials. Furthermore, we discuss how the thermodynamic quantities change as a function of the trap-aspect ratio and how the effective system dimension can be defined. In particular, we carefully analyse not only the thermodynamic limit but also the respective finite-size corrections similarly to a corresponding seminal study in 3D [15]. As so far photon BEC experiments have only been performed in an isotropic setup, this theoretical paper paves the way towards future experiments with strongly anisotropic harmonic trapping potentials. Such potentials can be achieved, for instance, by ellipsoidally grinding the mirrors which allows, however, only for traps with small anisotropies. Thus, in view of achieving stronger anisotropies it is more promising to use direct laser writing as a microstructuring technique, as it is then possible to create potential landscapes with spatial variations of the order of the wavelength of the photons [16][18].

This paper is organised as follows. Section 2 introduces the setting and provides an analytical expression for the thermodynamic potential of an ideal Bose gas at the dimensional crossover from 2D to 1D. Equipped with this, Sec. 3 specialises to the photon gas and derives expressions for the critical particle number as well as for the condensate fraction. Afterwards, the specific heat of the photon gas is discussed in Sec. 4 which is finally used to define the effective dimension of the system in Sec. 5.

2 Grand-Canonical Potential

At first we analyse the thermodynamic properties of an ideal Bose gas at the dimensional crossover between 2D and 1D. To this end we consider a two-dimensional harmonic trap for bosons, where the trapping frequency in y-direction can be altered. Thus, the energy levels are given by:

$$E_{jn}(\lambda) = \hbar \Omega \left( j + \lambda n + \frac{1 + \lambda}{2} \right),$$

where $\Omega$ stands for the trapping frequency in x-direction and $\lambda = \Omega_y/\Omega$ denotes the trap-aspect ratio. We remark, that for a fully isotropic 2D oscillator we have $\lambda = 1$, whereas the one-dimensional case is approached in the limit $\lambda \to \infty$. Intuitively, the gas can already be
considered to be effectively one dimensional, if the energy spacing in $y$-direction is larger than the thermal energy $k_B T$, which leads for the trap-aspect ratio to the condition

$$\lambda > \lambda_{1D}. \quad (2)$$

Here we define the effective one-dimensional trap-aspect ratio

$$\lambda_{1D} = \frac{k_B T}{\hbar \Omega}. \quad (3)$$

Taking into account the energy levels (1), we have with the chemical potential $\mu$, the inverse temperature $\beta = 1/(k_B T)$, and the degeneracy $g$ for the grand-canonical potential [19]

$$\Pi = -\frac{g}{\beta} \sum_{j,n=0}^{\infty} \sum_{k=1}^{\infty} \frac{e^{-\beta (E_{jn}(\lambda) - \mu)k}}{k}. \quad (4)$$

Performing the sum over $j$ allows us to write the potential $\Pi$ in the form of a dimensional expansion

$$\Pi = \Pi_{1D} + \Delta \Pi(\lambda). \quad (5)$$

Here the one-dimensional grand-canonical potential reads

$$\Pi_{1D} = -\frac{g}{\beta} \frac{\hbar \Omega}{b} I(\tilde{\mu}, b, -1), \quad (6)$$

where we introduced the dimensionless variables $b = \beta \hbar \Omega$ and $\tilde{\mu} = (1 + \lambda)/2 - \mu/(\hbar \Omega)$, cf. Ref. [20]. The correction $\Delta \Pi(\lambda)$ to the 1D potential, which takes the second dimension into account, depends on the trap-aspect ratio $\lambda$ via

$$\Delta \Pi(\lambda) = -\frac{g}{\beta} \frac{\hbar \Omega}{b} \sum_{n=1}^{\infty} I(\tilde{\mu} + \lambda n, b, -1). \quad (7)$$

Both in Eqs. (6) and (7) we used the auxiliary function

$$I(a, b, l) = \sum_{k=1}^{\infty} k^l \frac{e^{-abk}}{1 - e^{-bk}}. \quad (8)$$

In the case $l = 0$ the sum (8) can be analytically approximated for small $b$ via a cutoff regularisation, whereas in Ref. [20] a dimensional regularisation was performed. The respective analytical details are worked out in Appendix A. We find with the results (49) and (53) in Appendix A the following analytical form of the total grand-canonical potential:

$$\Pi = -\frac{g}{\beta} \frac{\hbar \Omega}{b} \left\{ f(\tilde{\mu}) + \frac{1}{b} \tilde{\xi}_2 \left( e^{-\tilde{\mu}b} \right) + \frac{1}{2} \tilde{\zeta}_1 \left( e^{-\tilde{\mu}b} \right) + \frac{b}{12} \tilde{\zeta}_0 \left( e^{-\tilde{\mu}b} \right) \right\}$$

$$-\frac{g}{\beta} \frac{\hbar \Omega}{b} \left\{ \frac{1}{\lambda b^2} \zeta_3 \left( e^{-(\tilde{\mu}+\lambda)b} \right) + \frac{1}{2 \lambda b} \zeta_2 \left( e^{-(\tilde{\mu}+\lambda)b} \right) + \frac{1}{12 \lambda} \zeta_1 \left( e^{-(\tilde{\mu}+\lambda)b} \right) + \frac{1}{\lambda} \int_{\tilde{\mu}+\lambda}^{\infty} dy \ f(y) \right\}$$

$$+\frac{1}{2} \left[ f(\tilde{\mu} + \lambda) + \frac{1}{b} \zeta_2 \left( e^{-(\tilde{\mu}+\lambda)b} \right) + \frac{1}{2} \zeta_1 \left( e^{-(\tilde{\mu}+\lambda)b} \right) + \frac{b}{12} \zeta_0 \left( e^{-(\tilde{\mu}+\lambda)b} \right) \right] + \ldots. \quad (9)$$
Here, $\zeta_l(x) = \sum_{k=1}^{\infty} x^k / k^l$ denotes the polylogarithm\cite{21} and $f(\bar{\mu})$ is defined in Eq. (50). The dots indicate here and in the following terms of order $b^2$ and higher. We remark, that the one-dimensional limit is given by $\lambda \to \infty$, which corresponds to the vanishing of the last two lines in Eq. (9).

In the following we discuss the thermodynamic consequences of the grand-canonical potential (9) for a general ideal Bose gas. But for illustrating the functional dependencies of the thermodynamic quantities we specialise these general results to the photon BEC experiments in Bonn\cite{12,22}. There, we have to take into account the two polarisational degrees of freedom of the photons resulting in the degeneracy $g = 2$. For typical values, i.e. $T_0 = 300$ K and $\Omega = 40$ GHz, the system can be considered to be effectively one dimensional, if the trap-aspect ratio fulfils condition (2) with $\lambda_{1D} \approx 156$. Moreover, since the photon BEC experiment is performed at room temperature $T_0$\cite{12,22}, the approximation of small $b$ is well fulfilled, as we have then $b_0 \approx 6 \times 10^{-3}$.

3 Particle Number

By calculating the derivative $N = -\partial \Pi / \partial \mu$, we find from the potential (9) for the total particle number

$$N = g \left\{ -\frac{1}{b} f'(\bar{\mu}) + \frac{1}{b} \zeta_1 \left( e^{-\bar{\mu}b} \right) + \frac{1}{2} \zeta_0 \left( e^{-\bar{\mu}b} \right) + \frac{b}{12} \zeta_{-1} \left( e^{-\bar{\mu}b} \right) + \frac{1}{\lambda b^2} \zeta_2 \left( e^{-(\bar{\mu}+\lambda)b} \right) + \frac{1}{2\lambda b} \zeta_1 \left( e^{-(\bar{\mu}+\lambda)b} \right) + \frac{1}{12\lambda} \zeta_0 \left( e^{-(\bar{\mu}+\lambda)b} \right) + \frac{1}{\lambda b} f(\bar{\mu}+\lambda) \right. \\
+ \frac{1}{2} \left[ -\frac{1}{b} f'(\bar{\mu}+\lambda) + \frac{1}{b} \zeta_1 \left( e^{-(\bar{\mu}+\lambda)b} \right) + \frac{1}{2} \zeta_0 \left( e^{-(\bar{\mu}+\lambda)b} \right) + \frac{b}{12} \zeta_{-1} \left( e^{-(\bar{\mu}+\lambda)b} \right) \right] \right\} + \ldots .$$

This explicit expression allows to determine both the critical particle number and the condensate fraction.

3.1 Critical Particle Number

In order to calculate the critical particle number, we consider the deep condensate limit $\bar{\mu} \to 0$. We remark, that this limit corresponds to the order parameter approach, as worked out in Ref. \cite{20}, where the ground-state particle number is used as an order parameter for the BEC phase transition and only the excited states are treated in a thermodynamic way. This approach corresponds to describing the Bose gas in the thermodynamic limit. In the present work, however, we treat all states, including the ground state, thermodynamically as this description is closer to the experimental situation, where the system is finite. With this the particle number (10) can be written in the form

$$N \approx N_0 + N_c,$$  \hspace{1cm} (11)

with the ground-state particle number

$$N_0 = \frac{g}{e^{\bar{\mu}b} - 1},$$  \hspace{1cm} (12)
which acquires in the limit $\bar{\mu} \to 0$ the form $N_0 \approx g/(\bar{\mu}b)$, and the critical particle number

$$N_c = g \frac{\gamma - \ln(b)}{b} + g \left\{ \frac{1}{\lambda b^2} \zeta_2 \left( e^{-\lambda b} \right) + \frac{1}{2\lambda b} \zeta_1 \left( e^{-\lambda b} \right) + \frac{1}{12\lambda} \zeta_0 \left( e^{-\lambda b} \right) + \frac{1}{\lambda b} f(\lambda) \right\} + \frac{1}{2} \left\{ \frac{1}{b} \zeta_1 \left( e^{-\lambda b} \right) + \frac{1}{2} \zeta_0 \left( e^{-\lambda b} \right) + \frac{b}{12} \zeta_{-1} \left( e^{-\lambda b} \right) \right\} + \mathcal{O} \left( (\bar{\mu}b)^0 \right),$$

(13)

with the Euler-Mascheroni constant $\gamma \approx 0.577$. Note that, due to the limit process involved, this result is only accurate up to order $\mathcal{O} \left( (\bar{\mu}b)^0 \right)$, but it is still accurate to all orders of $\lambda b$. Moreover, we note the same structure as for the grand-canonical potential in Eq. (5), namely the bare one-dimensional quantity in the first line gets modified by the terms in the other two lines, which depend on the trap-aspect ratio and describe the influence of the second dimension, see Fig. 1 a). Note that the first line in (13) follows from the first line in Eq. (10) by applying the Robinson formula [23],

$$\zeta_l \left( e^{-a} \right) = \frac{(-a)^{l-1}}{(l-1)!} \left\{ \sum_{k=1}^{l-1} \frac{1}{k} - \ln a \right\} + \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \zeta(l-k),$$

(14)

where $\zeta(l)$ denotes the Riemann-ζ function, in order to expand the occurring polylogarithms with positive integer index $l$ and by using for the corresponding polylogarithms with negative index the representation

$$\zeta_{-l} \left( e^{-a} \right) = \frac{1}{(1 - e^{-a})^{l+1}} \sum_{k=0}^{l-1} \left\langle \frac{l}{k} \right\rangle e^{-(l-k)a},$$

(15)

with the Eulerian numbers $\left\langle \frac{l}{k} \right\rangle$ [24].

From (13) we find for the 1D critical particle number

$$N_{c, 1D} = g \frac{k_B T}{\hbar \Omega} \left[ \gamma - \ln \left( \frac{\hbar \Omega}{k_B T} \right) \right].$$

(16)

Near two dimensions, where $b \lambda \ll 1$ holds, the critical particle number $N_{c, 1D}$ reads

$$N_{c, 2D} = N_{c, 1D} + g \left\{ \frac{\zeta(2)}{\lambda(h\Omega\beta)^2} + \frac{1}{2\hbar \Omega \beta} \left[ \ln(h\Omega\beta) + \frac{\ln\Gamma(\lambda) - \ln(2\pi \hbar \Omega \beta)}{\lambda} - \psi_0(\lambda) \right] \right\},$$

(17)

where $\psi_0(x)$ denotes the digamma function. In the strict 2D limit, i.e. $\lambda = 1$, this reduces to

$$N_{c, 2D} = N_{c, 1D} + g \left\{ \frac{k_B T}{\hbar \Omega} \right\}^2 + k_B T \frac{2\hbar \Omega}{\hbar \Omega} \left[ \gamma - \ln(2\pi) \right].$$

(18)

At first, a comparison with Ref. [20] shows that Eq. (16) is exact, whereas the corresponding expression for the two-dimensional critical particle number (18) contains the last term in addition. This difference is due to our approach, where we calculate at first the one-dimensional quantities and approximate afterwards the corresponding two-dimensional ones. As the leading order of the relative deviation of our result (18) compared to the corresponding one in
Figure 1: a) Critical particle number \( \langle 13 \rangle \) at room temperature \( T_0 \) for varying trap-aspect ratio (blue/solid line). The green (dashed dotted) line illustrates the 1D limit (16). b) Critical particle number \( \langle 13 \rangle \) for different trap-aspect ratios \( \lambda \) as a function of the temperature \( T \) normalised to the room temperature \( T_0 \). The blue (solid) line represents the isotropic 2D case, i.e. \( \lambda = 1 \), the orange (dashed) line is for \( \lambda = k_B T_0 / \hbar \Omega \equiv \lambda_{1D} \), and the green (dash-dotted) depicts the 1D limit, i.e. \( \lambda \to \infty \).

Refs. [20, 25] is of the order of the magnitude of \( b \) itself, the difference for the experimental parameter regime is of the order \( b_0 \approx 6 \times 10^{-3} \) and, thus, negligible for all practical purposes. Already here, we also encounter a qualitative difference between the two special cases of dimensions. In 1D the critical particle number \( \langle 13 \rangle \) depends linearly on the temperature, apart from the logarithmic term, whereas in 2D the leading order in Eq. (18) is quadratic in the temperature. In Fig. 1(b) we plot the critical particle number \( \langle 13 \rangle \) as a function of the temperature for different trap-aspect ratios \( \lambda \). Neither in the isotropic case \( \lambda = 1 \) nor in the 1D case, which amounts to the limit \( \lambda \to \infty \), the functional dependence of the critical particle number on the temperature changes. However, we note the different exponents one and two in accordance with (16) and (18), which can be interpreted as a sign for the corresponding dimension. For an intermediate trap-aspect ratio of \( \lambda = k_B T_0 / (\hbar \Omega) \equiv \lambda_{1D} \) the temperature dependence changes qualitatively for \( T \approx T_0 \). For smaller temperatures the curve coincides with the 1D curve, while for larger temperatures the orange curve gets parallel to the 2D curve. This means, that in the former case the system behaves effectively one dimensional, whereas in the latter case the system reveals a two-dimensional behaviour. This observation completely agrees with the criterion (2) for quasi one-dimensionality.

Finally, we also solve the critical particle number \( N(T_c) \) for the critical temperature \( T_c \) in the respective dimension. In 1D, we obtain from directly inverting Eq. (16) the implicit equation

\[
T_{c, \text{1D}} = \frac{\hbar \Omega}{g k_B} \left( \frac{N}{\gamma - \ln(\hbar \Omega / k_B T_{c, \text{1D}})} \right).
\]

(19)

In [3] it is derived, that no Bose-Einstein condensation is possible for a harmonic potential in one spatial dimension, as the critical temperature tends to zero in this limit. In contrast to that, we find in (19) a finite critical temperature and, thus, the possibility for a Bose-Einstein condensate. The difference between the approach in [3] and our approach is, that the former work is performed in the thermodynamic limit, whereas we assume a finite system size. Therefore, the divergent value \( \zeta(1) \), which occurs in [3] in the limit of a harmonic
trapping potential, is resolved in our case by the logarithm appearing in (19). In nearly two dimensions we find by iterating Eq. (17) once for $b\lambda \ll 1$

$$T_{c, \approx 2D} \approx \frac{\hbar \Omega}{k_B} \sqrt{\frac{\lambda N}{g\zeta(2)}} \left\{ 1 - \frac{\lambda}{4\zeta(2)} \sqrt{\frac{g\zeta(2)}{\lambda N}} \left[ 2\gamma - \psi_0(\lambda) + \ln \Gamma(\lambda) - \ln \left( \frac{2\pi \sqrt{g\zeta(2)/(\lambda N)}}{\lambda} \right) \right] + \frac{1}{2} \ln \left( \frac{g\zeta(2)}{\lambda N} \right) \right\}.$$  

(20)

In the strict 2D case $\lambda = 1$ we obtain

$$T_{c, 2D} = \frac{\hbar \Omega}{k_B} \sqrt{\frac{N}{g\zeta(2)}} \left\{ 1 - \frac{1}{4\zeta(2)} \sqrt{\frac{g\zeta(2)}{N}} \left[ 3\gamma - \ln \left( \frac{2\pi g\zeta(2)}{N} \right) \right] \right\}.$$  

(21)

Comparing Eqs. (19) and (21) we note the following two differences between the two limiting cases of the dimension. At first, in 1D the total particle number contributes with the exponent one to the critical temperature, whereas in 2D it appears with a square root. A second difference is the occurrence of the logarithm. In 1D the logarithm shows up already in the zeroth order of the expression, while in 2D the logarithm determines the first finite-size correction.

Furthermore, we remark that in the experimental situation of photon BECs the temperature is always fixed to the room temperature $T_0$. However, one can effectively vary the temperature for instance by changing the trapping frequency since $T/T_c = \Omega_c/\Omega$, which follows from the definition of $b$. A second possibility is to change the total particle number as was already used in the experiments of [22]. The drawback of this definition is, however, that only for integer dimensions $D = 1, 2$ an analytic correspondence is available in the form $T/T_c = \left( N_c/N \right)^{1/D}$. The latter also plays a role in the next section, when dealing with the condensate fraction.

### 3.2 Condensate Fraction

Now, we calculate the condensate fraction $N_0/N$ in the deep condensate limit, i.e. for $N \gg N_c$. Thus, using Eq. (11) we have

$$\frac{N_0}{N} \approx 1 - \frac{N_c}{N}. \quad \text{(22)}$$

In 1D we find for the fraction $N_c/N$ by using the critical particle number (16)

$$\left( \frac{N_c}{N} \right)_{1D} = \frac{T}{T_c} \left[ 1 - \frac{\ln(T/T_c)}{\gamma - \ln(h\Omega/k_BT_0)} \right], \quad \text{(23)}$$

so we have in leading order a linear temperature dependence. In contrast to this, when we approach two spatial dimensions, using the corresponding expression (17), we find in the leading order a quadratic dependence on the temperature

$$\left( \frac{N_c}{N} \right)_{2D} = \left( \frac{T}{T_c} \right)^2 + \frac{1}{2\zeta(2)} \sqrt{\frac{g\zeta(2)}{N}} \left\{ \frac{T}{T_c} - \left( \frac{T}{T_c} \right)^2 \right\}$$

$$\times \left[ -\frac{1}{2} \ln \frac{g\zeta(2)}{N} + \ln \Gamma(\lambda) - \ln \left( \frac{2\pi \sqrt{g\zeta(2)/N}}{\lambda} \right) - \psi_0(\lambda) + 2\gamma \right] + \frac{T}{T_c} \left( 1 + \frac{1}{\lambda} \right) \ln \frac{T}{T_c} \right\}.$$  

(24)
Figure 2: Condensate fraction $N_0/N$ for fixed particle number $N = 100,000$. The blue (solid) line represents the isotropic 2D case, i.e. $\lambda = 1$, the orange (dashed) line is for $\lambda = 500$, and the green (dash-dotted) line shows the 1D limit $\lambda \to \infty$. The red crosses are experimental values with the corresponding errors for the 2D case [26].

In the strict 2D limit Eq. (24) reduces to

$$
\left( \frac{N_c}{N} \right)_{2D} = \left( \frac{T}{T_c} \right)^2 + \sqrt{\frac{g\zeta(2)}{N}} \left\{ \left[ \frac{T}{T_c} \right] - \left( \frac{T}{T_c} \right)^2 \right\} \frac{3\gamma - \ln(2\pi g\zeta(2)/N)}{2\zeta(2)} - \frac{T\ln(T/T_c)}{T_c\zeta(2)}. \tag{25}
$$

Figure 2 shows a numerical calculation of the temperature dependence of the condensate fraction for an experimentally realistic number of $N = 100,000$ photons for different values of the trap-aspect ratio $\lambda$. The numerical calculation of the condensate fraction is done as follows. At first, we invert the particle number equation (10) in order to extract the dimensionless chemical potential $\tilde{\mu}$. We then use this value to calculate the ground-state population $N_0$ and, thus, the condensate fraction. We note that the isotropic 2D curve is in good agreement with the experiment of Ref. [26]. The discrepancy in the thermal phase is attributed to the finite resolution of the experimental apparatus. Moreover, we observe the inverted parabolic temperature dependence [25]. Also in the quasi 1D case the curve agrees with the linear temperature dependence predicted in [23]. For the curve with an intermediate trap-aspect ratio of $\lambda = 500$ the curve shows characteristics of both the 1D curve and as the temperature increases also of the 2D curve, meaning that here the effective dimension of the system changes from 1D to 2D.
4 Specific Heat

In the following we calculate the specific heat $C_N$ for a constant particle number $N$ from the internal energy $U = \Pi + TS + \mu N$ according to

$$C_N = \frac{\partial U}{\partial T} + \frac{\partial U}{\partial \mu} \left( \frac{\partial \mu}{\partial T} \right)_N,$$

where the second term takes the condition of a fixed particle number into account. However, due to the complexity of the resulting formula, we restrict the discussion to the deeply condensed case at first. In this approximation we find for the internal energy

$$U \approx E_0 (N_0 + N_c) + \hbar \Omega \lambda \Delta N + g \frac{\hbar \Omega}{b} \left[ \frac{f'(\lambda)}{2} - \frac{f(\lambda)}{\lambda} \right] + \hbar \Omega \left[ \frac{\zeta(2)}{b^2} - \frac{1}{2b} + \frac{2}{\lambda b^2} \zeta(3) \left( e^{-\lambda b} \right) \right].$$

We note, that in this limit we have $\partial \mu / \partial T \approx 0$, so the specific heat in the leading order of $1/b$ reads

$$C_N \approx g k_B \left[ \frac{2 \zeta(2)}{b} + \frac{6}{\lambda b^2} \zeta(3) \left( e^{-\lambda b} \right) + \frac{3\lambda + 1}{b} \zeta(2) \left( e^{-\lambda b} \right) \right].$$

Therefore, taking into account Eq. (19), in the one-dimensional limit the specific heat is given by

$$C_{N,1D} = N k_B \frac{2 \zeta(2)}{\gamma - \ln(h \Omega/k_B T_c)} \frac{T}{T_c},$$

and in leading order in $b$ near 2D we obtain from Eq. (20)

$$C_{N,\approx2D} \approx 6 \lambda N k_B \left( \frac{T}{T_c} \right)^2 \frac{\zeta(3)}{\zeta(2)} \left\{ 1 - \frac{1}{2 \zeta(2)} \sqrt{\frac{g \zeta(2)}{\lambda N}} \ln \Gamma(\lambda) - \ln \left( \frac{2\pi \sqrt{g \zeta(2)/\lambda N}}{N} \right) - \frac{1}{2} \ln \frac{g \zeta(2)}{\lambda N} \right\}. \quad (30)$$

In the strict 2D limit Eq. (30) reduces to

$$C_{N,2D} \approx 6 \lambda N k_B \left( \frac{T}{T_c} \right)^2 \frac{\zeta(3)}{\zeta(2)} \left\{ 1 - \frac{1}{2 \zeta(2)} \sqrt{\frac{g \zeta(2)}{N}} \right\}. \quad (31)$$

After, deriving these analytic formulas in the deeply condensed case, we discuss now the obtained results and compare them with the experimental results of Ref. [22]. In Fig. 3 we plot the full specific heat for different values of the trap-aspect ratio $\lambda$ as a function of the temperature $T$. In the isotropic 2D case we find the suspected $\lambda$-like transition with the high-temperature limit of $C_N = 2N k_B$, in accordance with the Dulong-Petit law. Note, that in our case the specific heat does not undergo a jump at the critical point as it occurs in the thermodynamic limit [20]. Instead, it remains a continuous function, as we are considering a
finite system. However, as we increase the trap-aspect ratio, we see that this characteristic transition vanishes and a plateau emerges just above the critical temperature. This plateau has the value $C_N = Nk_B$ and, thus, resembles the one-dimensional Dulong-Petit law, meaning that here the system, indeed, behaves as one-dimensional. By further increasing the temperature the system approaches again the 2D Dulong-Petit law. The reason for this peculiar behaviour is as follows. As soon as for a given trap-aspect ratio the thermal energy is large enough to also thermally occupy the squeezed dimension, cf. the condition (2), the system behaves again 2D. A similar behaviour is well known from the thermodynamics of molecular gases [19]. At low temperatures only the translational degrees of freedom of the molecules can be thermally excited and, thus, only those can contribute to the specific heat. Increasing the temperature above a certain threshold allows the molecules to rotate such that these degrees of freedom additionally contribute to the specific heat. Increasing the temperature even further allows also the vibrational modes of the molecules to be thermally excited.

We note the different behaviour of the specific heat in the low-temperature limit, which is worked out in Eqs. (29) and (31). Thus, in contrast to the condensate fraction and the critical particle number, using the specific heat can be instrumental to define and to determine the effective dimension of the system both in the low and the high temperature limit.

5  Phase Diagram and Effective Dimension

Finally, we analyse how the phase diagram of the system changes as a function of the trap aspect ratio $\lambda$ and the temperature $T$ by plotting the specific heat $C_N$ in Fig. (4) a). At first
we note that the phase transition from the BEC to the thermal phase happens at the critical temperature $T_c$, which is calculated by inverting the critical photon number (13) with the limiting cases (19) and (21). Moreover, in the thermal phase, we can directly read off the effective dimension of the system according to the respective Dulong-Petit law, as is explained at the end of Sec. 4. The dashed white line depicting the criterion (2) discriminates between the different dimensional behaviour. However, therefrom we can only learn about the effective dimension in the thermal phase. From Eqs. (29) and (31), though, we read off, that in the condensed regime the effective dimension follows from the polynomial dependency of the specific heat on the temperature. Therefore, we suggest to define as the effective dimension in the BEC phase the double-logarithmic derivative

$$d_{\text{BEC}} = -\frac{1}{Nk_B} \frac{\partial \ln C_N}{\partial \ln b}.$$  

Figure 4 b) shows the corresponding results as a function of both the temperature $T$ and the trap aspect ratio $\lambda$. We note that in the thermal phase this definition yields a constant value of 0 due to the Dulong-Petit law except right at the crossover from 2D to 1D. Thus, this definition cannot be used in the thermal case to determine the effective system dimension. In the BEC phase, however, we find values between 1 and 2 according to the limiting cases deduced from Eqs. (29) and (31). The precise value of the BEC dimension is determined by the temperature and trap-aspect ratio. We see that, for an increasing trap-aspect ratio, the system behaves, indeed, quasi one-dimensional.

Summarising the two observations from Fig. 4, we suggest to define the effective dimension of the system by

$$d = \begin{cases} 
C_N/(Nk_B), & \text{in the thermal phase,} \\
 d_{\text{BEC}}, & \text{in the BEC phase.}
\end{cases}$$  

With this we are able to describe the effective dimension of the system in both the Bose-condensed and the thermal regime. However, as Fig. 5 shows, the definition (32) fails in the
Figure 5: Effective dimension $d$ of the ideal Bose gas at the dimensional crossover as defined by (33) as a function of both the temperature and the trap-aspect ratio. The solid white curve shows the critical temperature obtained by inverting (13), whereas the dashed white line indicates the quasi-1D criterion (2). The calculation has been performed for $N = 1000$ particles. In the colourbars only the region of interest between 0 and 2 is labelled.

immediate vicinity of the phase boundary, as it turns out to be non-continuous. This can be read off from the reddish area in the plot. Nevertheless, we also note, that the effective dimension of the system changes from 2D to effective 1D in agreement with the criterion (2). We remark, that in the crossover region both the temperature and the trap-aspect ratio determine the effective dimension of the system.

6 Conclusions

In this paper we present an analytical description of the dimensional crossover from 1D to 2D for an ideal Bose gas in terms of a dimensional expansion, see Eq. (5). We find the same structure for all investigated thermodynamic quantities, such as the critical particle number, the condensate fraction, and the specific heat, namely that the 1D expression gets corrected by terms yielding the 2D result. Furthermore, from the specific heat we are able to define an effective dimension $d$, given by Eq. (33), in both the BEC and the thermal phase. This definition shows a change of the effective dimension, which is consistent with the criterion (2). But we also note, that this definition has a minor drawback as it fails near the phase
boundary, as can be seen in Fig. [5]. However, our results allow to determine the effective
dimension of the system for a given temperature and trap-aspect ratio. We especially focus
on how to determine the effective dimension by examining the exponent in the BEC case and
by observing the Dulong-Petit law in the thermal regime. We remark, that our calculational
approach, which is based on an expansion in the smallness parameter \( \hbar \Omega / (k_B T) \), is especially
suitable for photon gases, where this value is of the order of a few per mille.

The present work could be extended to also determine the spatio-temporal behaviour of the
correlation function of the ideal Bose gas at the dimensional crossover, which has already been
measured for an isotropic two-dimensional photon gas [27]. Concerning the fact, that in the
2D photon BEC experiments a retarded thermo-optic interaction is dominant, despite of an
additional negligible contact interaction [12,28,29], it is an interesting question, whether this
is still true in the quasi-1D case. Moreover, for a more realistic modelling of the experiments,
one needs to include also the pump and the decay processes, as a photon gas in a dye-filled
microcavity is intrinsically an open system. Another research direction would be to investigate
in view of the dimensional crossover different potential landscapes, such as potentials with
arbitrary exponents, c.f. [3], or even anharmonic potentials [30].

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A  Cutoff Regularisation

The aim of this appendix is to work out the behaviour of the auxiliary functions \( I(a, b, l) \)
defined in Eq. (8) for integer \( l \) and also to provide a procedure allowing to approximate these
functions analytically. First we start with two recursion relations obeyed by the auxiliary
functions. For increasing the integer \( l \) we have

\[
I(a, b, l + 1) = -\frac{1}{b} \frac{\partial}{\partial a} I(a, b, l),
\]

whereas decreasing \( l \) yields correspondingly

\[
I(a, b, l - 1) = b \int_{a}^{x} dx \ I(x, b, l).
\]

Thus, from the analytical knowledge of one particular \( I(a, b, l^*) \) all other functions \( I(a, b, l) \)
can be calculated analytically.
A.1 Special Case $l^* = 0$

It turns out, that the case $l^* = 0$ can be calculated analytically for small values of $b$. According to Eq. (8) we start with

$$I(a, b, 0) = \sum_{k=1}^{\infty} \frac{e^{-abk}}{1 - e^{-bk}}. \quad (36)$$

In order to calculate expression (36), we follow Ref. [20] and perform an expansion for small values of $b$. However, the first step is to include also the $k = 0$ term in the summation (36). As this is a divergent term, we add and subtract the first three terms of the corresponding Laurent series. Note, that in Ref. [20] only the first term of the Laurent series is introduced yielding an approximation up to $O(b^0)$. However, here we need higher order terms for obtaining a converging result for the two-dimensional case. Thus, we have

$$I(a, b, 0) = \sum_{k=0}^{\infty} e^{-abk} \left( \frac{1}{1 - e^{-bk}} - \frac{1}{bk} - \frac{1}{2} \frac{bk}{12} \right) + \sum_{k=1}^{\infty} e^{-abk} \left( \frac{1}{bk} + \frac{1}{2} \frac{bk}{12} \right) + O(b^2). \quad (37)$$

In the first term we replace the summation by an integral using the Euler-Maclaurin formula

$$\sum_{n=0}^{N} f(n) \approx \int_{0}^{N} dn f(n) + \frac{1}{2} [f(0) + f(N)]. \quad (38)$$

Due to the construction of expression (37), all higher terms in the Euler-Maclaurin series (38) vanish exactly. In the second term we recognise the polylogarithmic functions $\zeta_n(x)$ with $n = -1, 0, +1$. Thus, we have

$$I(a, b, 0) = \int_{0}^{\infty} dk e^{-abk} \left( \frac{1}{1 - e^{-bk}} - \frac{1}{bk} - \frac{1}{2} \frac{bk}{12} \right) + \frac{1}{b} \zeta_1(e^{-ab}) + \frac{1}{2} \zeta_0(e^{-ab}) + \frac{b}{12} \zeta_{-1}(e^{-ab}) + O(b^2). \quad (39)$$

Whereas in Ref. [20] the remaining integrals are solved by using a dimensional regularisation, we introduce here an infrared cutoff $\Lambda$ as the integrands are divergent for $k \to 0$:

$$I(a, b, 0) = \lim_{\Lambda \to 0} \int_{\Lambda}^{\infty} dk e^{-abk} \left( \frac{1}{1 - e^{-bk}} - \frac{1}{bk} - \frac{1}{2} \frac{bk}{12} \right) + \frac{1}{b} \zeta_1(e^{-ab}) + \frac{1}{2} \zeta_0(e^{-ab}) + \frac{b}{12} \zeta_{-1}(e^{-ab}) + O(b^2). \quad (40)$$

First, we obtain

$$\int_{\Lambda}^{\infty} dk (bk)^n e^{-abk} = \frac{1}{a^{n+1}b} \Gamma(n + 1, ab\Lambda),$$

where $\Gamma(s, x)$ is the upper incomplete $\Gamma$ function. For small $\Lambda$ we find in leading order

$$\Gamma(0, ab\Lambda) \approx -\frac{\gamma - \ln(ab\Lambda)}{b}, \quad (42)$$
whereas the incomplete \( \Gamma \) functions with indices \( n \geq 1 \) simply reduce to the standard \( \Gamma \) functions:

\[
\Gamma(n, ab\Lambda) \approx \Gamma(n), \quad n \geq 1.
\] (43)

In the remaining first integral of Eq. (40) we substitute \( x = e^{-bk} \) and calculate by using the incomplete beta function,

\[
B(x; a, b) = \int_0^x dt \ t^{a-1}(1-t)^{b-1},
\] (44)

the integral

\[
\int_\Lambda^\infty dk \ \frac{e^{-abk}}{1-e^{-bk}} = \frac{1}{b} B(e^{b\Lambda}; a, 0).
\] (45)

This yields in the limit of small \( \Lambda \)

\[
B(e^{-b\Lambda}; a, 0) \approx -\ln(b\Lambda) - \gamma - \psi_0(a).
\] (46)

Inserting Eqs. (42), (43) and (46) into Eq. (40) we finally have

\[
I(a, b, 0) = \frac{1}{b} \left[ \ln(a) - \psi_0(a) - \frac{1}{2a} - \frac{1}{12a^2} \right] + \frac{1}{b} \zeta_1 \left( e^{-ab} \right) + \frac{1}{2} \zeta_0 \left( e^{-ab} \right) + \frac{b}{12} \zeta_{-1} \left( e^{-ab} \right) + \mathcal{O}(b^2),
\] (47)

which coincides with the result in Ref. [20], apart from the additional higher order terms. In the following we calculate \( I(a, b, -1) \) from applying the recurrence relation (35). To this end we use the Stirling formula [21],

\[
\ln \Gamma(z) \approx z(\ln z - 1) - \frac{1}{2} \ln(2\pi z),
\] (48)

for regularising the upper integration limit in Eq. (35), and obtain

\[
I(a, b, -1) = f(a) + \frac{1}{b} \zeta_2 \left( e^{-ab} \right) + \frac{1}{2} \zeta_1 \left( e^{-ab} \right) + \frac{b}{12} \zeta_0 \left( e^{-ab} \right) + \mathcal{O}(b^2),
\] (49)

where we defined

\[
f(a) = \frac{1}{2} \ln \left( \frac{a}{2\pi} \right) - a \left[ \ln(a) - 1 \right] + \ln \Gamma(a) - \frac{1}{12a}.
\] (50)

This result is still correct to order \( \mathcal{O}(b^2) \), because the recurrence relation (35) preserves the corresponding order.

A.2 Resummation for Second Dimension

In [7] we have seen, that we also need to calculate a sum over the auxiliary functions (8). With the result (49) we can also analytically approximate the sum

\[
S = \sum_{n=1}^{\infty} I(a + \lambda n, b, -1)
\] (51)
by using again the Euler-Maclaurin series \([38]\). Thus, we obtain the approximation

\[
S \approx \int_{1}^{\infty} dn \ I(a + \lambda n, b, -1) + \frac{1}{2} I(a + \lambda, b, -1). \tag{52}
\]

Taking Eq. \([49]\) into account, we have

\[
S = \frac{1}{\lambda b^2} \zeta_3 \left( e^{-(a+\lambda)b} \right) + \frac{1}{2\lambda b} \zeta_2 \left( e^{-(a+\lambda)b} \right) + \frac{1}{12\lambda} \zeta_1 \left( e^{-(a+\lambda)b} \right) + \frac{1}{\lambda} \int_{a+\lambda}^{\infty} dy \ f(y)
+ \frac{1}{2} I(a + \lambda, b, -1) + \mathcal{O}(b^2). \tag{53}
\]

We note that the error, stemming from the Euler-Maclaurin approximation in Eq. \([52]\), cannot be evaluated in a systematic way. However, we show in the next section, that the performed approximation yields errors, which are small in the relevant parameter regime of photon gases.

### A.3 Analytical vs. Numerical Summation

Finally, we compare the analytical results from the preceding sections with a numerical summation of Eq. \([36]\) itself. Figure 6 shows the relative error of the numerical approximation \([47]\) with respect to the direct numerical evaluation of the sum \([36]\). The orange line shows our result, whereas the blue line shows the accuracy achieved in Ref. \([20]\). At first, we note that both results yield a good approximation as \(b\) tends to 0. However, as we use additional terms from the Laurent series in Eq. \([37]\), the accuracy of our result is increased compared to the result from Ref. \([20]\).
In order to analyse the error of the 2D result we first note that the sum (51) can also be performed by using the definition (36) and interchanging the summation signs, which yields

\[ S_{2D} = \sum_{k=1}^{\infty} \frac{e^{-abk}}{k(1 - e^{-bk})(e^{bk} - 1)}. \] (54)

Note, that due to the factor \(1/k\) this expression cannot be treated analytically along the philosophy of Ref. [20] and this appendix. However, expression (54) can be used as a numerical comparison with the analytical approximation obtained in (53). The relative error of the approximation of the two-dimensional sum (51) is shown in Fig. 7. It reveals, as suspected, the same overall behaviour as the approximation of the one-dimensional sum, namely that the approximation gets better at small \(b\) and worse at large values of \(b\).

References


